Homoclinic and heteroclinic symmetry-breaking controller sets for Bogdanov-Takens singularity

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Abstract

This is a supplementary file that presents the proofs for [6, theorems 2.5, 2.9, and 2.10].

An efficient nonlinear time transformation method has been recently developed and applied for global bifurcation varieties of homoclinic and heteroclinic varieties of codimension two singularities [1, 2, 7]. This is an efficient alternative approach to the classical use of Melnikov functions; *e.g.,* see [9, 10]. Both approaches have been usually applied using one-small scaling variable. Since all parameters are scaled using one parameter, the approach typically lead to a one-dimensional transition variety and fits well within a codimension-two singularity. Transition varieties must have a dimension of three in order that they would partition the parameter space in four dimensions. Although the scaling constants play a role in accommodating the higher dimensional transitions sets (*e.g.,* see [8]), we include three scaling parameters $\epsilon_1, \epsilon_2, \epsilon_3$. We derive an estimation for controller sets for homoclinic and heteroclinic bifurcations. Our symbolic estimations are accurate enough for many control engineering applications. Higher order approximations than our derived formulas are also feasible, but it is beyond the scope of this paper; *e.g.,* see [1, 2, 7, 8] for highly accurate one- and two-dimensional transition varieties. Symbolic estimations for these bifurcations are useful for an efficient management of the nearby oscillating dynamics.

Theorem 0.1. *[6, Theorem 2.5] When $a_2 > 0$ and $\mu_0 = \mathscr{O}(|\mu_1|^2)$, the bifurcated limit cycles disappear via two distinct quaternary homoclinic controller sets estimated by*

$$T_{HmC\pm} := \left\{ (\mu_0, \mu_1, \mu_2, \mu_3) |\, \mu_2 = \frac{8b_2}{5a_2}\mu_1 \pm \frac{9\sqrt{2}\pi}{32}\mu_3\sqrt{-\mu_1} \mp \frac{9\sqrt{2}\pi}{32}\frac{\mu_0}{\sqrt{-\mu_1}} + \sqrt{-\mu_1}\mathscr{O}(|\mu_1|, |\frac{{\mu_0}^2}{{\mu_1}^3}|) \right\}. \tag{0.1}$$

The leading estimated terms for the homoclinic cycles $\Gamma_\pm$ give rise to an effective criteria for the magnitude control of the nearby oscillating dynamics. These are given by

$$(x(\varphi), y(\varphi)) = \left(-\frac{\sin^2(\varphi)\cos(\varphi)\sqrt{3-cos(2\varphi)}}{\sqrt{a_2}}\mu_1, \pm\frac{\sqrt{2}}{2}\sqrt{-\mu_1}(\cos(2\varphi) - 1) \right) + (\mathscr{O}(|\mu_1|^{\frac{3}{2}}), \mathscr{O}(|\mu_1|)), \quad for \varphi \in [0, \pi].$$

Proof. We apply a nonlinear time transformation method and include multiple scaling parameters ϵ_i for $i = 1, 2, 3$; see [2, 7, 8]. Namely, we use the rescaling transformations

$$\begin{aligned} x = {\epsilon_1}^2\tilde{x}, y = \epsilon_1\tilde{y}, t = {\epsilon_1}^{-1}\tilde{t}, \mu_0 = {\epsilon_1}^3\left(\gamma_1 + \gamma_{01}\epsilon_1 + \gamma_{02}\epsilon_2\right), \mu_1 = {\epsilon_1}^2\left(\gamma_2 + \epsilon_1\gamma_{11} + \epsilon_2\gamma_{12} + \epsilon_3\gamma_{13}\right), \\ \mu_2 = {\epsilon_1}^2\gamma_{21} + \epsilon_1\epsilon_2\gamma_{22} + \epsilon_1\epsilon_3\gamma_{23} + \epsilon_1\mathscr{O}(|(\epsilon_1, \epsilon_2, \epsilon_3)|^2), \mu_3 = \epsilon_1\gamma_{31} + \epsilon_2\gamma_{32} + \epsilon_3\gamma_{33} + \epsilon_1\epsilon_2\gamma_{34}. \end{aligned} \tag{0.2}$$

These transform the differential system given in [6, equations (1.3)] into

$$\begin{aligned} \dot{\tilde{x}} &= \gamma_1\left(\gamma_2\epsilon_1 + \gamma_{02}\epsilon_2\right) + \gamma_2\left(1 + \epsilon_1\gamma_{11} + \epsilon_2\gamma_{12} + \epsilon_3\gamma_{13}\right)\tilde{y} + \left(\epsilon_1\gamma_{21} + \epsilon_2\gamma_{22} + \epsilon_3\gamma_{23}\right)\tilde{x} + a_2\tilde{y}^3 \\ &\quad + \left(\epsilon_1\gamma_{31} + \epsilon_2\gamma_{32} + \epsilon_3\gamma_{33} + \epsilon_1\epsilon_2\gamma_{34}\right)\tilde{x}\tilde{y} + \epsilon_1 b_2\tilde{x}\tilde{y}^2, \\ \dot{\tilde{y}} &= -\tilde{x} + \left(\epsilon_1\gamma_{21} + \epsilon_2\gamma_{22} + \epsilon_3\gamma_{23}\right)\tilde{y} + \left(\epsilon_1\gamma_{31} + \epsilon_2\gamma_{32} + \epsilon_3\gamma_{33} + \epsilon_1\epsilon_2\gamma_{34}\right)\tilde{y}^2 + \epsilon_1 b_2\tilde{y}^3. \end{aligned} \tag{0.3}$$

The unperturbed system, *i.e.,* when $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3) = \mathbf{0}$, is a Hamiltonian system with Hamiltonian $H = \gamma_1\tilde{y} + \frac{1}{2}\tilde{x}^2 + \frac{1}{2}\gamma_2\tilde{y}^2 + \frac{1}{4}a_2\tilde{y}^4$. We further Taylor-expand the new state variables and a time-rescaling transformation in terms of the scaling parameters ϵ_i for $i = 1, 2, 3$ as

$$\begin{aligned} \tilde{x}(\varphi) := \tilde{x}_0(\varphi) + \sum {\epsilon_j}^i\tilde{x}_{ij}(\varphi), \quad \tilde{y}(\varphi) := \tilde{y}_0(\varphi) + \sum {\epsilon_j}^i\left(p_{ij}\cos(2\varphi) + q_{ij}\right), \\ \tilde{t} = \Phi\tau, \quad \Phi := \phi_0 + \sum {\epsilon_j}^i\phi_{ij}, \end{aligned} \tag{0.4}$$

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where the sum $\sum$ without indices stands for the double sum $\sum_{i=1}^{\infty}\sum_{j=1}^{3}$ and $\varphi \in [0,\pi]$. Let $\gamma_1 := 0$, $\gamma_{01} := 0$, $\gamma_{02} := 0$, $\gamma_2 := -1$, $\gamma_{34} := 1$. Then, Hamiltonian of the unperturbed system holds a homoclinic cycle that connects the stable and unstable manifolds of the origin, *i.e.,* the homoclinic orbit follows $H(\tilde{x},\tilde{y}) = 0$. When the rescaling variables ϵ_i for $i = 1,2,3$ becomes non-zero, the homoclinic cycle still holds for a homoclinic variety of codimension-one in the parameter space. The idea of the nonlinear time transformation method is to iteratively calculate the homoclinic cycle and homoclinic variety in terms of powers of ϵ_i. We here only deal with zero and first order approximations, *i.e.,* (p_0, q_0, x_0, ϕ_0) and $(p_{1j}, q_{1j}, x_{1j}, \phi_{1,j})$ for $j = 1,2,3$. We remark that there is only a homoclinic cycle for system (0.3). However, this will turn out to be two homoclinic cycles $\Gamma_\pm$ for [6, equations (1.3)], depending on the sign of ϵ_1 in (0.9). The zero order approximation is given by $(\tilde{x}_0(\varphi), \tilde{y}_0(\varphi))$, where we assume that

$$\tilde{y}_0 := p_0\cos(2\varphi) + q_0 \quad \text{and} \quad \tilde{x}_0(0) = \tilde{x}_0(\tfrac{\pi}{2}) = 0. \tag{0.5}$$

Hence, $(\tilde{y}_0(0), \tilde{y}_0(\frac{\pi}{2})) = (p_0 + q_0, q_0 - p_0)$. Since Hamiltonian is constant over the homoclinic cycle, we have $H(\tilde{x}_0(\frac{\pi}{2}), \tilde{y}_0(\frac{\pi}{2})) = H(0, p_0 + q_0)$. Furthermore, $\frac{\partial H}{\partial \tilde{y}}(0, p_0 + q_0) = 0$ due to the fact that $(\tilde{x}_0(0), \tilde{y}_0(0))$ is an equilibrium for the unperturbed Hamiltonian system. These equations give rise to

$$\begin{gathered} p_0 = \tfrac{\sqrt{2}}{2\sqrt{a_2}}, \quad q_0 = -\tfrac{\sqrt{2}}{2\sqrt{a_2}}, \qquad \tilde{y}_0 = \tfrac{\sqrt{2}}{2\sqrt{a_2}}\cos(2\varphi) - \tfrac{\sqrt{2}}{2\sqrt{a_2}}, \\ \tilde{x}_0 = \pm\tfrac{\sin^2(\varphi)\cos(\varphi)\sqrt{3-cos(2\varphi)}}{\sqrt{a_2}}, \quad \text{and} \quad \phi_0(\varphi) := -\tfrac{\tilde{x}_0}{\tilde{y}_0'}. \end{gathered}$$

Let $q_{1j} := 0$ for $j = 1,2,3$. Then, $\tilde{y}_{11} = p_{11}\cos(2\varphi)$, $\tilde{y}_{12} = p_{12}\cos(2\varphi)$, $\tilde{y}_{13} = p_{13}\cos(2\varphi)$ and the first-order approximation follows

$$\tilde{x} = \tilde{x}_0 + \epsilon_1\tilde{x}_{11} + \epsilon_2\tilde{x}_{12} + \epsilon_3\tilde{x}_{13}, \quad \tilde{y} = \tilde{y}_0 + \epsilon_1 p_{11}\cos(2\varphi) + \epsilon_2 p_{12}\cos(2\varphi) + \epsilon_3 p_{13}\cos(2\varphi). \tag{0.6}$$

Next, the terms of the first-order in terms of ϵ_i for $i = 1,2,3$ in $\phi\dot{x}$ give rise to

$$\begin{gathered} \phi_0 x_{11}' + p_{11}\cos(2\varphi) - x_0\gamma_{21} - x_0 y_0\gamma_{31} - y_0\gamma_{11} - \gamma_{01} - 3a_2{y_0}^2 p_{11}\cos(2\varphi) + \phi_{11}x_0' - b_2 x_0 {y_0}^2 = 0, \\ \phi_0 x_{1i}' + p_{1i}\cos(2\varphi) - x_0\gamma_{3(i+1)} - x_0 y_0\gamma_{4(i+1)} - y_0\gamma_{2(i+1)} \\ -\gamma_{1(i+1)} - 3a_2{y_0}^2 p_{1i}\cos(2\varphi) + \phi_{1i}x_0' = 0, \end{gathered} \tag{0.7}$$

see also [7, Equation 2.22a and 2.22b]. Now consider the first-order ϵ_i-terms in $\phi\dot{y}$ along with equations (0.7). By eliminating ϕ_{1j}-terms from these equations, an integrating factor and an integration, similar to the proof of [7, Equation 2.30], we derive

$$\begin{gathered} \int_0^\varphi y_0'\big(y_0\gamma_{11} - p_{12}\cos(2\varphi) + x_0\gamma_{21} + y_0 x_0\gamma_{31} + b_2 x_0 {y_0}^2 + \gamma_{01} + 3a_2{y_0}^2 p_{12}\cos(2\varphi)\big)d\varphi \\ +x_0 x_{11} + y_{11}g(y_0) + \int_0^\varphi x_0'\left({y_0}^2\gamma_{31} + \gamma_{21}y_0 + b_2{y_0}^3\right)d\varphi = 0, \\ \int_0^\varphi y_0'\big(y_0\gamma_{2(i+1)} - p_{1i}\cos(2\varphi) + x_0\gamma_{3(i+1)} + y_0 x_0\gamma_{4(i+1)} + \gamma_{1(i+1)} + 3a_2{y_0}^2 p_{1i}\cos(2\varphi)\big)d\varphi \\ +x_0 x_{1i} + y_{1i}g(y_0) + \int_0^\varphi x_0'\left({y_0}^2\gamma_{4(i+1)} + \gamma_{3(i+1)}y_0\right)d\varphi = 0, \text{ for } i = 2,3. \end{gathered} \tag{0.8}$$

Evaluating equation (0.7) at $\varphi = \pi$ and (0.8) at $\varphi = \pi, \pi/2$, we obtain nine number of linear equations. These give rise to the scaling parameters

$$\begin{gathered} \gamma_{21} = -\tfrac{8b_2}{5a_2} + \tfrac{9\sqrt{2}\pi}{32}\gamma_{31}, \quad \gamma_{22} = \tfrac{9\sqrt{2}\pi}{32}\gamma_{32}, \quad \gamma_{23} = \tfrac{9\sqrt{2}\pi}{32}\gamma_{33}, \\ \gamma_{11} = 0, \quad \gamma_{12} = 0, \quad \gamma_{13} = 0, \quad \epsilon_1 := \pm\sqrt{-\mu_1}. \end{gathered} \tag{0.9}$$

Finally, we substitute these into the equation for μ_2 in (0.2) and derive transition sets $T_{HmC\pm}$ given in equation (0.1). □

Theorem 0.2. *[6, Theorem 2.9] Let $|\mu_0| = \mathscr{O}(|\mu_1|^2)$. For the case $a_2 < 0$, there is a heteroclinic cycle Λ. This connects the equilibrium E_+ with the saddle E_-. The corresponding heteroclinic bifurcation occurs at the heteroclinic controller set approximated by*

$$T_{HtC} := \left\{(\mu_0,\mu_1,\mu_2,\mu_3)|\, \mu_2 = \frac{2b_2}{5a_2}\mu_1 + \frac{9}{16}\mu_3\sqrt{\mu_1} - \frac{9}{16}\frac{\mu_0}{\sqrt{\mu_1}} + \mathscr{O}(||(\mu_1,\mu_3)||^{\frac{3}{2}})\right\} \quad \textit{and} \quad a_2 < 0. \tag{0.10}$$

The most leading estimated terms for Λ are

$$(x,y) = \left(\mu_1 \tfrac{\sqrt{2}}{2}\sin(2\varphi)\sqrt{a_2\cos^2(2\varphi)+2+a_2}, \tfrac{\sqrt{\mu_1}}{\sqrt{-a_2}}\cos(2\varphi)\right) + (\mathscr{O}(|\mu_1|^{\frac{3}{2}}), \mathscr{O}(|\mu_1|)).$$

Proof. Here, we use the rescaling transformations (0.2) and (0.4) when $\gamma_1 := 0$, $\gamma_{02} := 0$, $\gamma_2 := 1$, $\gamma_{11} := \gamma_{12} := \gamma_{13} := 0$, and $\gamma_{34} := 1$. The unperturbed heteroclinic orbit connects the two saddles $\left(0, \pm\frac{1}{\sqrt{-a_2}}\right)$. We assume that equations (0.5) hold. Similar to [7, Equation 2.11], we have

$$y_0(0) = -\tfrac{1}{\sqrt{-a_2}} = p_0 + q_0, y_0(\tfrac{\pi}{2}) = \tfrac{1}{\sqrt{-a_2}} = q_0 - p_0, \text{ and thus,}$$
$$\tilde{y}_0 = p_0\cos(2\varphi) + q_0 = \tfrac{1}{\sqrt{-a_2}}\cos(2\varphi), q_0 = 0.$$

We compute the unperturbed heteroclinic orbit via $H(\tilde{x}_0, \tilde{y}_0) = H(0, \pm(-a_2)^{\frac{-1}{2}})$ where $H(\tilde{x},\tilde{y}) = \frac{1}{2}\tilde{x}^2 - \frac{1}{2}\tilde{y}^2 + \frac{1}{4}a_2\tilde{y}^4$. Therefore, $\tilde{x}_0 = \frac{\sqrt{2}}{2}\sin(2\varphi)\sqrt{a_2\cos^2(2\varphi)+2+a_2}$. The first-order terms for $i = 1,2,3$ in $\phi\dot{x}$ follow equations (0.7) and equations (0.8) hold. We need the first-order terms in $\phi\dot{y}$ given by (see [7, Equations 2.22a and 2.22b])

$$\phi_0 y_{11}' - \gamma_{21}y_0 - {y_0}^2\gamma_{31} - b_2{y_0}^3 + \phi_{11}y_0' = 0, \phi_0 y_{1i}' - \gamma_{3(i+1)}y_0 - {y_0}^2\gamma_{4(i+1)} + \phi_{1i}y_0' = 0, \tag{0.11}$$

for $i = 2,3$. We evaluate equations (0.7) and (0.11) at $\varphi = 0, \frac{\pi}{2}$, while equations (0.8) are computed at $\varphi = \pi/2$. These lead to fifteen linear equations and $\gamma_{21} = \frac{2b_2}{5a_2} + \frac{9}{16}\gamma_{31}$, $\gamma_{32} = \frac{16}{9}\gamma_{22}$, and $\gamma_{33} = \frac{16}{9}\gamma_{23}$. Thus, transition varieties (0.10) are derived by substitution of these values into the rescaling transformation for μ_2. □

Theorem 0.3. *[6, Theorem 2.10] Let $a_2 < 0$. There are two homoclinic cycles $\Lambda_\pm$ connecting the stable and unstable manifolds of $E_\pm$ at the estimated controller sets*

$$T^{\pm}_{HmC} := \left\{(\mu_0,\mu_1,\mu_2,\mu_3)|\, \mu_1 = \tfrac{10^{\frac{2}{3}}}{(-a_2)^{\frac{5}{2}}}{\mu_0}^{\frac{2}{3}} - \tfrac{49.19204541}{(-a_2)^{\frac{5}{2}}}b_2\mu_0 - \tfrac{8.203865604}{10^{\frac{-2}{3}}(-a_2)^{\frac{11}{6}}}\mu_2{\mu_0}^{\frac{1}{3}} + \tfrac{4.355675048}{10^{\frac{-2}{3}}(-a_2)^3}\mu_3{\mu_0}^{\frac{2}{3}}\right\}. \tag{0.12}$$

The homoclinic Λ_+ occurs when $\mu_0 > 0$ while Λ_- corresponds with negative values of μ_0. The leading estimated terms for (x,y)-coordinates of the homoclinic cycles $\Lambda_\pm$ are

$$\mp 0.7157063998\,(\cos(2\varphi) - 0.2299428741) \quad \textit{and} \quad \mp\mu_1 0.3622053022\sqrt{2}\sin(2\varphi)\sqrt{cos(2\varphi) + 2.28512403471548},$$

for $\varphi \in [0,\pi]$, respectively. This is useful for the management of its nearby oscillating dynamics.

Proof. We first use transformations $x = (-a_2)^{\frac{3}{2}}\hat{x}$, $y = (-a_2)^{\frac{1}{2}}y$, and time rescaling $t = -\frac{1}{a_2}\tau$ to change the coefficient a_2 to -1. Then, we have

$$\dot{\hat{x}} = (-a_2)^{\frac{-5}{2}}\mu_0 + (-a_2)^{-2}\mu_1\hat{y} + (-a_2)^{-1}\mu_2\hat{x} - \hat{y}^3 + (-a_2)^{\frac{-1}{2}}\mu_3\hat{x}\hat{y} + b_2\hat{x}\hat{y}^2, \tag{0.13}$$
$$\dot{\hat{y}} = -\hat{x} + (-a_2)^{-1}\mu_2\hat{y} + (-a_2)^{\frac{-1}{2}}\mu_3\hat{y}^2 + b_2\hat{y}^3.$$

Let $\mu_0^* := (-a_2)^{\frac{-5}{2}}\mu_0$, $\mu_1^* := (-a_2)^{-2}\mu_1$, $\mu_2^* := (-a_2)^{-1}\mu_2$, $\mu_3^* := (-a_2)^{\frac{-1}{2}}\mu_3$. Next, we replace μ_i^* with μ_i and $\hat{y}$ with $\tilde{y}$ for simplicity. Now apply the rescaling transformations (0.2) and expansion (0.4) when

$$\gamma_1 = \pm 0.1, \gamma_{01} = \gamma_{02} = 0, \gamma_2 = -1, \gamma_{11} = \gamma_{13} = 0, \gamma_{12} = 1, \gamma_{21} = \gamma_{22} = 0, \gamma_{34} = 0. \tag{0.14}$$

Recall that the unperturbed system is Hamiltonian and it holds a homoclinic cycle Λ_+ for $\gamma_1 > 0$. This connect the stable and unstable manifolds of the saddle E_+. The homoclinic cycle Λ_- happens for $\gamma_1 < 0$ and corresponds with E_-. Following the proof of Theorem 0.1, we apply equations (0.5), $(\tilde{y}_0(0), \tilde{y}_0(\frac{\pi}{2})) = (p_0 + q_0, q_0 - p_0)$, $H(\tilde{x}_0(\frac{\pi}{2}), \tilde{y}_0(\frac{\pi}{2})) = H(0, p_0 + q_0)$, and $\frac{\partial H}{\partial \tilde{y}}(0, p_0 + q_0) = 0$ to obtain

$$\tilde{y}_0 = \mp 0.7157063998\cos(2\varphi) \mp 0.2299428741,\ p_0 = \mp 0.7157063998,\ q_0 = \mp 0.2299428741.$$

where $\tilde{x}_0$ is obtained from $H(\tilde{x}_0, \tilde{y}_0) = \frac{1}{2}{\tilde{x}_0}^2 - \frac{1}{2}{\tilde{y}_0}^2 + \frac{1}{4}a_2{\tilde{y}_0}^4 \pm \frac{1}{10}\tilde{y}_0 = H(0, p_0 + q_0)$. Here, equations (0.7) and (0.8) hold and we evaluate them at $\varphi = \pi$ and $\varphi = \pi, \pi/2$, respectively. We obtain $\gamma_{23} = \pm 0.2464356892\gamma_{33}$ and $\gamma_{31} := \pm 1.129378222 b_2$. A substitution into the rescaling transformations completes the proof. □

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