High order term ideals and a one-parametric normal form family of first integrals

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Abstract

In this draft, we discuss the finite determinacy and universal unfolding for the steady-state bifurcations of two truncated Hopf-zero normal form families. These are the most generic cases obtained in [1]. We prove that the simplest truncated orbital normal forms admit a family of first integrals. As a consequence, a family of invariant manifolds arises for the differential normal form system.

We first deal with its finite determinacy of the steady-state solutions of the reduced planar differential system and, then, present a one-parametric family of first integrals for the normal form system associated with case $s < r$. Next, we do the same for the family associated with $r = s$ in Proposition 0.5 and Theorem 0.3.

We here discuss what is the suitable truncation degree. This brings up the jet sufficiency problem for normal forms computations. We employ results from singularity theory [2, Definition 7.1, Proposition 1.4, Theorem 7.2 and Theorem 7.4] to derive such jet sufficiency results for the most generic normal form systems, *i.e.,* $s = 1$ and arbitrary $r \geq 1$. Let $\mathscr{G}$ be either the identity group or the group generated by Γ where $\Gamma : (x, \rho, \theta) \mapsto (x, -\rho, \theta)$. The differential system

$$\begin{aligned} \dot{x} &= \mu_1 + 2\rho^2 + b_s x^{s+1} + \sum_{k=s+1}^{2s} b_k x^{k+1} + \sum_{i=2}^{s} \mu_i x^{i-1} + \sum_{i=0}^{s-1} \mu_{s+2+i} x^{i+1}, \qquad (0.1)\\ \dot{\rho} &= \frac{b_s}{2} x^s \rho + \sum_{k=s+1}^{2s} \frac{b_k}{2} x^k \rho - \sum_{i=2}^{s} \frac{(i-1)}{2} \mu_i x^{i-2} \rho + \frac{1}{2} \sum_{i=0}^{s-1} \mu_{s+2+i} x^i \rho, \end{aligned}$$

is $\mathscr{G}$-equivariant. Then, we recall a strong $\mathscr{G}$-equivalence relation from [2, page 166] as follows. We say that G and H are strong $\mathscr{G}$-equivalent if and only if there are invertible matrix $S(x, \rho, \lambda)$ and invertible coordinate changes $(x, \rho, \lambda) \mapsto (X(x, \rho, \lambda), \varrho(x, \rho, \lambda), \lambda)$ so that $G(x, \rho, \lambda) = S(x, \rho, \lambda) H\left(X(x, \rho, \lambda), \varrho(x, \rho, \lambda), \lambda\right).$ Here, $X(0, 0, 0) = 0$, $\varrho(0, 0, 0) = 0$, and for all $g \in \mathscr{G}$,

$$X(x, g\rho, \lambda) = gX(x, \rho, \lambda),\ \varrho(x, g\rho, \lambda) = g\varrho(x, \rho, \lambda),\ S(x, g\rho, \lambda)g = gS(x, \rho, \lambda)$$

while $S(0, 0, 0), (dX)_{0,0,0}$, and $(d\varrho)_{0,0,0}$ are scalar multiples of the identity matrix.

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Denote $\mathscr{E}_{x,\rho,\lambda}$ for the local ring of all $\mathscr{G}$-invariant smooth germs in (x,ρ,λ)-variables. Further, the unique maximal ideal $\mathcal{M}$ in $\mathscr{E}_{x,\rho,\lambda}$ is generated by $\mathcal{M} :=< x,\rho,\lambda >$. Define $\overrightarrow{\mathcal{M}}$ as a $\mathscr{E}_{x,\rho,\lambda}$-module generated by

$$\overrightarrow{\mathcal{M}}^k := \left\langle \begin{pmatrix} x^{k-i-j}\rho^j\lambda^i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^{k-i-j}\rho^j\lambda^i \end{pmatrix} \Big| \, 0 \leq i,j, \text{ and } i+j \leq k \right\rangle \subset \overrightarrow{\mathscr{E}}_{x,\rho,\lambda}(\mathscr{G}). \tag{0.2}$$

Here, $\overrightarrow{\mathscr{E}}_{x,\rho,\lambda}(\mathscr{G})$ stands for the smooth $\mathscr{G}$-equivariant germ vector fields. The ideal $\overrightarrow{\mathcal{M}}$ includes all $\mathscr{G}$-equivariant flat vector fields and $\overrightarrow{\mathcal{M}}^k$ (for $k \in \mathbb{N}$) constitutes a decreasing chain sequence of $\mathscr{E}_{x,\rho,\lambda}$-modules, *i.e.*, $\overrightarrow{\mathcal{M}}^{k+1} \subsetneq \overrightarrow{\mathcal{M}}^k$.

Theorem 0.1 (Cases $(s=1, r>1)$)**.** *Let $b_1 \neq 0$. Then, the map $G(x,\rho,\lambda) = \left(\lambda + 2\rho^2 + b_1x^2, \frac{b_1}{2}x\rho\right)$ is strongly $\mathscr{G}$-equivalent to $G+p$ for any $p \in \overrightarrow{\mathcal{M}}^3$. In other words, $\overrightarrow{\mathcal{M}}^3 \subseteq \mathscr{P}(G,\mathscr{G})$. In particular, the steady-state solutions are two determined.*

Proof. Recall that $\mathscr{E}_{x,\rho,\lambda}$-module $\mathcal{K}_s(\mathscr{G})$ is generated by $\mathcal{M}\binom{G_1}{0}$, $\mathcal{M}\binom{0}{G_1}$, $\mathcal{M}\binom{G_2}{0}$, $\mathcal{M}\binom{0}{G_2}$, $\mathcal{M}^2\binom{G_{1x}}{G_{2x}}$, and $\mathcal{M}^2\binom{G_{1\rho}}{G_{2\rho}}$. We claim that $\overrightarrow{\mathcal{M}}^3 \subseteq \mathcal{K}_s(\mathscr{G})$. Thereby for any $p \in \overrightarrow{\mathcal{M}}^3 \subseteq \mathcal{K}_s(\mathscr{G})$, $G+p$ is strongly $\mathscr{G}$-equivalent with G. The latter claim follows from [2, Theorem 7.2, page 205]. Indeed, the higher order term module $\mathscr{P}(G,\mathscr{G})$ includes the intrinsic $\mathscr{G}$-equivariant $\mathscr{E}_{x,y,\lambda}$-submodule of $\mathcal{K}_s(\mathscr{G})$. For any $\mathscr{G}$-equivariant $\mathscr{E}_{x,\rho,\lambda}$-module $\overrightarrow{\mathscr{I}}$, the inclusion $\overrightarrow{\mathscr{I}} \subseteq \mathcal{K}_s(\mathscr{G})$ is equivalent with the inclusion $\overrightarrow{\mathscr{I}} \subseteq \mathcal{K}_s(\mathscr{G}) + \mathcal{M}\overrightarrow{\mathscr{I}}$. This is the basis of our arguments in the proof for here and also for that of Theorem 0.3. We introduce an equivalence relation $\simeq$ on $\overrightarrow{\mathscr{E}}_{x,\rho,\lambda}(\mathscr{G})$ defined by

$$\text{for every } A,B \in \overrightarrow{\mathscr{E}}_{x,\rho,\lambda}(\mathscr{G}), \quad A \simeq B \quad \text{ if and only if } \quad A-B \in \mathcal{M}\overrightarrow{\mathscr{I}}. \tag{0.3}$$

Hence, we let $\overrightarrow{\mathscr{I}} := \overrightarrow{\mathcal{M}}^3$ and $b_1 := 2$ without loss of generality. Thus, $\mathcal{M}\overrightarrow{\mathscr{I}} = \overrightarrow{\mathcal{M}}^4$. Further, let $\mathscr{G}$ be the identity group; the proof for $\mathscr{G} := \langle \Gamma \rangle$ readily requires some minor modifications. Therefore, $\mathcal{K}_s(\mathscr{G})$ includes vector field germs generated by

$$\rho\begin{pmatrix} G_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x\rho^2 \\ 0 \end{pmatrix}, \quad x\begin{pmatrix} G_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \rho x^2 \\ 0 \end{pmatrix}, \quad \lambda\begin{pmatrix} G_2 \\ 0 \end{pmatrix} = \begin{pmatrix} x\rho\lambda \\ 0 \end{pmatrix},$$
$$\rho\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x\rho^2 \end{pmatrix}, \quad x\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0, \\ \rho x^2 \end{pmatrix}, \quad \text{and} \quad \lambda\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x\rho\lambda \end{pmatrix}.$$

Furthermore, we have $x^2\begin{pmatrix} G_1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \lambda x^2 \\ 0 \end{pmatrix}$, $x\lambda\begin{pmatrix} G_1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} x\lambda^2 \\ 0 \end{pmatrix}$, $x^2\begin{pmatrix} G_{1x} \\ G_{2x} \end{pmatrix} - x\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \begin{pmatrix} 4x^3 \\ 0 \end{pmatrix}$, and $\rho^2\begin{pmatrix} G_{1x} \\ G_{2x} \end{pmatrix} - 4\rho\begin{pmatrix} G_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \rho^3 \end{pmatrix}$, while

$$\rho^2\begin{pmatrix} G_{1\rho} \\ G_{2\rho} \end{pmatrix} - \rho\begin{pmatrix} 0 \\ G_2 \end{pmatrix} = \begin{pmatrix} 4\rho^3 \\ 0 \end{pmatrix}, \rho^2\begin{pmatrix} G_1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \lambda\rho^2 \\ 0 \end{pmatrix}, \rho\lambda\begin{pmatrix} G_1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \rho\lambda^2 \\ 0 \end{pmatrix}, \text{ and } \lambda^2\begin{pmatrix} G_1 \\ 0 \end{pmatrix} \simeq \begin{pmatrix} \lambda^3 \\ 0 \end{pmatrix}.$$

Therefore, all these terms belong to $\mathcal{K}_s(\mathscr{G}) + \mathcal{M}\overrightarrow{\mathscr{I}}$. Now we consider

$$\rho^2\begin{pmatrix} 0 \\ G_1 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \rho^2\lambda \end{pmatrix}, \quad \rho\lambda\begin{pmatrix} 0 \\ G_1 \end{pmatrix} \simeq \begin{pmatrix} 0 \\ \rho\lambda^2 \end{pmatrix}, \quad \lambda^2\begin{pmatrix} G_{1\rho} \\ G_{2\rho} \end{pmatrix} - 4\rho\lambda\begin{pmatrix} G_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4x\lambda^2 \end{pmatrix},$$

and $x^2\begin{pmatrix}G_{1\rho}\\ G_{2\rho}\end{pmatrix}-4x\begin{pmatrix}G_2\\ 0\end{pmatrix}=\begin{pmatrix}0\\ x^3\end{pmatrix}$. These also conclude that $\begin{pmatrix}0\\ \rho^2\lambda\end{pmatrix}$, $\begin{pmatrix}0\\ \rho\lambda^2\end{pmatrix}$, $\begin{pmatrix}0\\ x\lambda^2\end{pmatrix}$, and $\begin{pmatrix}0\\ x^3\end{pmatrix}$ belong to $\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}}$. Next, recall that $\begin{pmatrix}x\rho\lambda\\ 0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$ and on the other hand, we have $\lambda x\begin{pmatrix}G_{1\rho}\\ G_{2\rho}\end{pmatrix}=\begin{pmatrix}4\lambda x\rho\\ \frac{1}{2}\lambda x^2\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. These imply that $\begin{pmatrix}0\\ \lambda x^2\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. Further, $\lambda^2\begin{pmatrix}0\\ G_1\end{pmatrix}\simeq\begin{pmatrix}0\\ \lambda^3\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}}$. Therefore, we have proved that all generators of $\overrightarrow{\mathscr{I}}$ are included in $\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}}$. This completes the proof by Nakayama lemma. □

Proposition 0.2 (Normal form first integrals for cases $s<r$)**.** *Consider a $s+1$-degree truncation of the planar differential system* (0.1) *coupled with $\dot\theta=1+\sum_{i=1}^{s}(c_i+\omega_i)x^i$ and let $\mu_i=0$ for all $2\leq i\leq 2s+1$. This system admits a family of normal form first integrals given by*

$$I_s(x,\rho,\mu_1):=\tfrac{1}{2(s+1)}\rho^{-2-2s}\left(\tfrac{2(s+1)}{s}\rho^2+\mu_1+b_sx^{s+1}\right). \tag{0.4}$$

Proof. We have

$$\dot x=\mu_1+2\rho^2+b_sx^{s+1},\quad \dot\rho=\tfrac{1}{2}b_sx^s\rho,\quad \dot\theta=1+\textstyle\sum_{i=1}^{s}(\gamma_i+\omega_i)x^i.$$

Thereby, we have

$$Mdx+Nd\rho:=\tfrac{1}{2}b_sx^s\rho dx-(\mu_1+2\rho^2+b_sx^{s+1})\,d\rho=0. \tag{0.5}$$

Next, we employ an integrating factor derived by

$$\mu=\exp\left(\textstyle\int-\frac{2}{\rho}\left(s+\frac{3}{2}\right)d\rho\right)=\rho^{-3-2s},\ \text{where}\ \frac{M_\rho-N_x}{-M}=-\tfrac{2}{\rho}\left(s+\tfrac{3}{2}\right).$$

This leads to

$$\frac{1}{2}b_sx^s\rho^{-2-2s}dx-\left(\mu_1+2\rho^2+b_sx^{s+1}\right)\rho^{-3-2s}d\rho=0.$$

Solving this first order equation gives rise to the first integral given by $I_s(x,\rho,\mu_1)$ in equation (0.4). □

Theorem 0.3 (Case $r=s=1$)**.** *Let $G(x,\rho,\lambda)=(G_1,G_2):=\left(\lambda+a_1x^2+2\rho^2+b_1x^2,-a_1x\rho+\dfrac{1}{2}b_1x\rho\right)$ for $a_1\neq-b_1$ and $2a_1\neq b_1$. Then, $\overrightarrow{\mathcal{M}}^3\subseteq\mathscr{P}(G,\mathscr{G})$. Besides, G is strongly $\mathscr{G}$-equivalent to $G+p$ for any $p\in\overrightarrow{\mathcal{M}}^3$. Assume that F is a smooth function and its two jet is the same as G. Then, the steady-state analysis of any system associated with F is two determined.*

Proof. Recall that $\mathcal{K}_s(\mathscr{G})\subseteq\mathscr{P}(G,\mathscr{G})$. Thus, it suffices to show that $\overrightarrow{\mathcal{M}}^3\subseteq\mathcal{K}_s(\mathscr{G})$. Let $\overrightarrow{\mathscr{I}}=\overrightarrow{\mathcal{M}}^3$ and apply the equivalence relation $\simeq$, that is defined on $\overrightarrow{\mathscr{E}}_{x,\rho,\lambda}(\mathscr{G})$ by equation (0.3). Hence, $A\simeq B$ if and only if $A-B\in\overrightarrow{\mathcal{M}}^4$. Then,

$$\lambda^2\begin{pmatrix}G_1\\ 0\end{pmatrix}\simeq\begin{pmatrix}\lambda^3\\ 0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}},\qquad x^2\begin{pmatrix}G_1\\ 0\end{pmatrix}\simeq\begin{pmatrix}\lambda x^2\\ 0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}},$$

and $\rho^2\begin{pmatrix}G_1\\ 0\end{pmatrix}\simeq\begin{pmatrix}\lambda\rho^2\\ 0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}}$ while due to $b_1-2a_1\neq0$,

$$\frac{2x}{b_1-2a_1}\begin{pmatrix}0\\ G_2\end{pmatrix}=\begin{pmatrix}0\\ x^2\rho\end{pmatrix}\in\mathcal{K}_s(\mathscr{G}),\qquad \frac{2\rho}{b_1-2a_1}\begin{pmatrix}0\\ G_2\end{pmatrix}=\begin{pmatrix}0\\ x\rho^2\end{pmatrix}\in\mathcal{K}_s(\mathscr{G}),$$

and $\lambda\begin{pmatrix}0\\G_2\end{pmatrix}=\begin{pmatrix}0\\\left(\frac{1}{2}b_1-a_1\right)\lambda x\rho\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. Since $x^2\begin{pmatrix}G_{1x}\\G_{2x}\end{pmatrix}-x\begin{pmatrix}0\\G_2\end{pmatrix}=2(a_1+b_1)\begin{pmatrix}x^3\\0\end{pmatrix}$ and $a_1+b_1\neq 0$, $\begin{pmatrix}x^3\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. Moreover, we have $\rho^2\begin{pmatrix}G_{1\rho}\\G_{2\rho}\end{pmatrix}-\rho\begin{pmatrix}0\\G_2\end{pmatrix}=\begin{pmatrix}4\rho^3\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$, and $x\rho\begin{pmatrix}G_{1\rho}\\G_{2\rho}\end{pmatrix}-x\begin{pmatrix}0\\G_2\end{pmatrix}=\begin{pmatrix}4x\rho^2\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. The later deduces that $\begin{pmatrix}0\\\frac{b_1-2a_1}{2}\rho^3\end{pmatrix}=\rho^2\begin{pmatrix}G_{1x}\\G_{2x}\end{pmatrix}-2(a_1+b_1)\begin{pmatrix}x\rho^2\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. On the other hand,

$$x\lambda\begin{pmatrix}G_1\\0\end{pmatrix}\simeq\begin{pmatrix}x\lambda^2\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\overrightarrow{\mathcal{M}}^4,\qquad \lambda\rho\begin{pmatrix}G_1\\0\end{pmatrix}\simeq\begin{pmatrix}\lambda^2\rho\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\overrightarrow{\mathcal{M}}^4,$$

and $x\rho\binom{G_1}{0}\simeq\binom{\lambda x\rho}{0}\in\mathcal{K}_s(\mathscr{G})+\overrightarrow{\mathcal{M}}^4$. However, for the cases of $\begin{pmatrix}x^2\rho\\0\end{pmatrix}$ and $\begin{pmatrix}0\\x^3\end{pmatrix}$, we have $\frac{x}{2(a_1+b_1)}\rho\begin{pmatrix}G_{1x}\\G_{2x}\end{pmatrix}-\frac{\rho}{2(a_1+b_1)}\begin{pmatrix}0\\G_2\end{pmatrix}=\begin{pmatrix}x^2\rho\\0\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$ and $\frac{2x^2}{b_1-2a_1}\begin{pmatrix}G_{1\rho}\\G_{2\rho}\end{pmatrix}-\frac{2}{b_1-2a_1}\begin{pmatrix}4x^2\rho\\0\end{pmatrix}=\begin{pmatrix}0\\x^3\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. Further, we have

$$\lambda x\begin{pmatrix}G_{1\rho}\\G_{2\rho}\end{pmatrix}-4\lambda\rho\begin{pmatrix}G_1\\0\end{pmatrix}\simeq\frac{b_1-2a_1}{2}\begin{pmatrix}0\\x^2\lambda\end{pmatrix},\quad \lambda^2\begin{pmatrix}G_{1\rho}\\G_{2\rho}\end{pmatrix}-4x\rho\begin{pmatrix}G_1\\0\end{pmatrix}\simeq\frac{b_1-2a_1}{2}\begin{pmatrix}0\\x\lambda^2\end{pmatrix}$$

and $\lambda\rho\begin{pmatrix}G_{1x}\\G_{2x}\end{pmatrix}-2(a_1+b_1)x\rho\begin{pmatrix}G_1\\0\end{pmatrix}\simeq\frac{b_1-2a_1}{2}\begin{pmatrix}0\\\lambda\rho^2\end{pmatrix}$. Thereby, $\begin{pmatrix}0\\x\lambda^2\end{pmatrix},\begin{pmatrix}0\\x^2\lambda\end{pmatrix}$, and $\begin{pmatrix}0\\\lambda\rho^2\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})+\overrightarrow{\mathcal{M}}^4$. Now the remaining vector field generators for $\overrightarrow{\mathcal{M}}^3$ are $\begin{pmatrix}0\\\lambda^3\end{pmatrix}$ and $\begin{pmatrix}0\\\rho\lambda^2\end{pmatrix}$. Since $\begin{pmatrix}0\\x\lambda^2\end{pmatrix}\in\mathcal{K}_s$ and $\lambda^2\begin{pmatrix}G_{1x}\\G_{2x}\end{pmatrix}=\begin{pmatrix}\lambda^2x\\\lambda^2\rho\end{pmatrix}\in\mathcal{K}_s$, these express that $\begin{pmatrix}0\\\rho\lambda^2\end{pmatrix}\in\mathcal{K}_s$. Then, $\lambda^2\begin{pmatrix}0\\G_1\end{pmatrix}\simeq\begin{pmatrix}0\\\lambda^3\end{pmatrix}\in\mathcal{K}_s(\mathscr{G})$. These conclude that all vector field generators of $\overrightarrow{\mathcal{M}}^3$ are included in $\mathcal{K}_s(\mathscr{G})+\mathcal{M}\overrightarrow{\mathscr{I}}$. Finally, by the Nakayama lemma, we have $\overrightarrow{\mathscr{I}}\subseteq\mathcal{K}_s(\mathscr{G})$ and this completes the proof. □

Theorem 0.4. *Let $\mathscr{G}:=\langle\Gamma\rangle$. Then, the equivariant tangent space for bifurcation problem*

$$F:\,(f_1,f_2)=\left(\lambda+a_1x^2+2\rho^2+b_1x^2,-a_1x\rho+\frac{1}{2}b_1x\rho\right)\tag{0.6}$$

is given by $T(F,\mathscr{G})=\overrightarrow{\mathcal{M}}^2(\mathscr{G})+\mathbb{R}\binom{2(a_1+b_1)x}{(\frac{b_1}{2}-a_1)\rho}+\mathscr{E}_\lambda\binom{1}{0}$ for $a_1\neq -b_1$ and $b_1\neq 2a_1$. Thus, the universal unfolding for (f_1,f_2) with respect to strong equivalence relation is given by $(G_1,G_2)=(f_1+\eta x, f_2-\frac{\eta\rho}{2})$.

Proof. Consider $F=(p,q\rho):=[p,q]$ where $v:=\rho^2$ and $u:=x$. Then, $RT(F,Z_2)$ is generated by [2, Table 3.1, Page 177] as follows

$$\operatorname{span}\{[p,0],[vq,0],[0,p],[0,q],[up_u,uq_u],[vp_u,vq_u],[\lambda p_u,\lambda q_u],[vp_v,vq_v]\}.\tag{0.7}$$

We apply Nakayama Lemma to prove that $\overrightarrow{\mathcal{M}}^2(\mathscr{G})\subseteq RT(F,Z_2)$. Hence, we need to show that $\overrightarrow{\mathcal{M}}^2(\mathscr{G})\subseteq RT(F,Z_2)+\overrightarrow{\mathcal{M}}^3(\mathscr{G})$. We have

$$\frac{1}{2(a_1+b_1)}[up_u,uq_u-q]=\begin{pmatrix}x^2\\0\end{pmatrix},\quad \frac{1}{2}[vp_v,vq_v]=\begin{pmatrix}\rho^2\\0\end{pmatrix},\quad \lambda[p,0]=\begin{pmatrix}\lambda^2+(a_1+b_1)\lambda x^2+2\lambda\rho^2\\0\end{pmatrix},$$

and these lead to the conclusion that

$$\begin{pmatrix}x^2\\0\end{pmatrix},\begin{pmatrix}\rho^2\\0\end{pmatrix},\begin{pmatrix}\lambda^2\\0\end{pmatrix}\in RT(F,Z_2)+\overrightarrow{\mathcal{M}}^3(\mathscr{G}).$$

Further, we have $\frac{2}{b_1-2a_1}[0,q] = \binom{0}{x\rho} \in RT(F, Z_2)$ and

$$\left[\frac{2}{b_1-2a_1}vp_u, \frac{2}{b_1-2a_1}vq_u - \frac{8(a_1+b_1)}{(b_1-2a_1)^2}p\right] = \begin{pmatrix}0\\ \rho^3\end{pmatrix} \in RT(F, Z_2).$$

Since $x[p,0] \in RT(F, Z_2)$, $\binom{\lambda x}{0} \in RT(F, Z_2) + \overrightarrow{\mathcal{M}}^3(\mathscr{G})$. Hence, the equality

$$[\lambda p_u, \lambda q_u] = \begin{pmatrix}2(a_1+b_1)x\lambda\\ (\frac{1}{2}b_1 - a_1)\lambda\rho\end{pmatrix} \in RT(F, Z_2) \tag{0.8}$$

implies that $\binom{0}{\lambda\rho} \in RT(F, Z_2) + \overrightarrow{\mathcal{M}}^3(\mathscr{G})$. On the other hand, $\overrightarrow{\mathcal{M}}^2(\mathscr{G})$ is generated by

$$\left\langle \begin{pmatrix}x^2\\ 0\end{pmatrix}, \begin{pmatrix}\rho^2\\ 0\end{pmatrix}, \begin{pmatrix}\lambda^2\\ 0\end{pmatrix}, \begin{pmatrix}x\lambda\\ 0\end{pmatrix}, \begin{pmatrix}0\\ x\rho\end{pmatrix}, \begin{pmatrix}0\\ \rho^3\end{pmatrix}, \begin{pmatrix}0\\ \lambda\rho\end{pmatrix} \right\rangle.$$

Hence, $\overrightarrow{\mathcal{M}}^2(\mathscr{G}) \subset RT(F, \mathscr{G})$. This and $[p,0] = \binom{\lambda + a_1x^2 + 2\rho^2 + b_1x^2}{0} \in RT(F, \mathscr{G})$ imply that

$$RT(F, \mathscr{G}) = \overrightarrow{\mathcal{M}}^2(\mathscr{G}) + \mathscr{E}_{x,\lambda}\begin{pmatrix}\lambda\\ 0\end{pmatrix}.$$

By [2, Page 212 and 213], the corresponding equivariant tangent space follows

$$T(F, \mathscr{G}) \quad = \quad RT(F, \mathscr{G}) + \mathbb{R}\{[p_u, q_u]\} + \mathscr{E}_\lambda\{[p_\lambda, q_\lambda]\}.$$

Here, $[p_u, q_u] = [2(a_1+b_1)x, \frac{b_1}{2} - a_1] = \binom{2(a_1+b_1)x}{(\frac{b_1}{2}-a_1)\rho}$ and $[p_\lambda, q_\lambda] = [1, 0] = \binom{1}{0}$. Thus,

$$T(F, \mathscr{G}) \quad = \quad RT(F, \mathscr{G}) + \mathscr{E}_\lambda\begin{pmatrix}1\\ 0\end{pmatrix} + \mathbb{R}\begin{pmatrix}2(a_1+b_1)x\\ (\frac{b_1}{2} - a_1)\rho\end{pmatrix} = \overrightarrow{\mathcal{M}}^2(\mathscr{G}) + \mathscr{E}_\lambda\begin{pmatrix}1\\ 0\end{pmatrix} + \mathbb{R}\begin{pmatrix}2(a_1+b_1)x\\ (\frac{b_1}{2} - a_1)\rho\end{pmatrix}. \tag{0.9}$$

Since $b_1 \neq 0$, we have $T(F, \mathscr{G}) \oplus \mathbb{R}\binom{x}{-\frac{1}{2}\rho} = \overrightarrow{\mathscr{E}}(\mathscr{G})$. This completes the proof. □

Proposition 0.5 (Normal form first integrals for cases $r = s$)**.** *Consider the $r+1$-truncation of the normal form differential system given by*

$$\begin{aligned}
\dot{x} \quad &= \quad 2\rho^2 + a_r x^{r+1} + b_r x^{r+1} + \sum_{k=r+1}^{2r} a_k x^{k+1} + \sum_{i=1}^{r} \mu_i x^{i-1} + \sum_{i=0}^{r-1} \mu_{i+r+1} x^{i+1}, \\
\dot{\rho} \quad &= \quad \frac{1}{2} b_r x^r \rho - \frac{a_r(r+1)}{2} x^r \rho - \sum_{k=r+1}^{2r} \frac{a_k(k+1)}{2} x^k \rho - \sum_{i=1}^{r} \frac{(i-1)}{2} \mu_i x^{i-2} \rho + \frac{1}{2} \sum_{i=0}^{r-1} \mu_{r+1+i} x^i \rho,
\end{aligned} \tag{0.10}$$

where the parameters $\mu_i = 0$ for $i \neq 1$. Then, this system admits a one-parametric family of normal form first integrals given by

$$I_r(x, \rho) := \tfrac{a_r r + a_r - b_r}{2a_r r + b_r r + 2a_r} \rho^{\frac{2(2a_r r + b_r r + 2a_r)}{a_r r + a_r - b_r}} - \tfrac{a_r r + a_r - b_r}{2(r+1)(a_r + b_r)} \mu_1 \rho^{\frac{2(r+1)(a_r + b_r)}{a_r r + a_r - b_r}} - \tfrac{a_r r + a_r - b_r}{2(r+1)} x^{r+1} \rho^{\frac{2(r+1)(a_r + b_r)}{a_r r + a_r - b_r}}. \tag{0.11}$$

Proof. The proof is similar to the proof of Proposition 0.2 and is omitted for briefness. □

References

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