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Bifurcation Control and Universal Unfolding for Hopf-Zero Singularities with Leading Solenoidal Terms*

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Dedicated to Professor James Murdock on the occasion of his 70th birthday

Abstract. In this paper we introduce *universal asymptotic unfolding normal forms* for nonlinear singular systems. Next, we propose an approach to finding the parameters of a parametric singular system that play the role of the universal unfolding parameters. These parameters *effectively* influence the local dynamics of the system. We propose a systematic approach to locating local bifurcations in terms of these parameters. Here we apply the proposed approach on Hopf-zero singularities whose first few low degree terms are incompressible. In this direction, we obtain novel orbital and parametric normal form results for such families by assuming a nonzero quadratic condition. Moreover, we give a truncated universal asymptotic unfolding normal form and prove the finite determinacy of the steady-state bifurcations for the two most generic subfamilies of the associated *amplitude* systems. We analyze the local primary bifurcations of equilibria and limit cycles, as well as the secondary Hopf bifurcations of invariant tori. The results are successfully implemented and verified using Maple. By employing the proposed approach, we design an effective multiple-parametric quadratic state feedback controller for a singular system on a three dimensional central manifold with two imaginary uncontrollable modes. We illustrate how our program systematically identifies the distinguished (universal unfolding) parameters, derives the estimated transition varieties in terms of these parameters, and locates the local primary and secondary bifurcations of equilibria, limit cycles, and invariant tori. This approach is useful in designing efficient nonlinear feedback controllers (single or multiple inputs) for local bifurcation control in engineering problems.

Key words. bifurcation control, universal asymptotic unfolding, primary and secondary bifurcations, Hopf-zero singularity, solenoidal vector fields

AMS subject classifications. 34C20, 34A34

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1. Introduction.

Any small perturbation of a singular differential system may substantially change the qualitative dynamics of the system. Therefore, for such engineering singular problems, a practical approach is to design a controller to inhibit its real world dynamics rather than being dominated by the mathematical modeling imperfections. This can be achieved by computing the *universal unfolding* of the differential system and can be used for a *bifurcation controller design*. Bifurcation control has many applications in engineering problems such as power, electronics, and mechanical systems, including predicting and preventing voltage

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collapse and oscillation in power networks, high-performance circuits, and oscillator designs; see, e.g., [9, 10]. The idea here is to design a controller for a nonlinear system so that the system follows a certain bifurcation branch and thus behaves as desired. Recently, normal form theory has been used for local bifurcation control; see [9, 24, 26, 27, 28, 29]. In this paper we describe how parametric normal form theory can propose effective feedback controller designs for an engineering problem. We apply it to Hopf-zero singularity, i.e.,

$$\dot{x} := f(x, y, z), \quad \dot{y} := z + g(x, y, z), \quad \dot{z} := -y + h(x, y, z), \tag{1.1}$$

for $(x, y, z) \in \mathbb{R}^3$, where f, g, h do not have linear and constant terms. Additionally, we assume that certain first few low degree terms of (1.1) constitute a solenoidal vector field (that is, the case I in (3.2)). Here a solenoidal vector field refers to a volume-preserving vector field. Throughout this paper, we will interchangeably use the terms *vector field*, denoted by $f\frac{\partial}{\partial x} + (z + g)\frac{\partial}{\partial y} + (-y + h)\frac{\partial}{\partial z}$ or $(f, z + g, -y + h)$, and *differential system* (1.1).

Normal form computations of vector fields cannot be performed up to an infinite degree Taylor expansion, and thus truncation at a finite degree is unavoidable. When the truncated normal form and the original system have the same *qualitative properties*, the system is called *finitely determined*; see [37, page vi]. We recall that *qualitative properties* are defined as invariances of a given equivalence relation. Thus, a system is finitely determined when it is equivalent to one of its truncated normal forms; see, e.g., [31, fold bifurcation (3.1), Lemma 3.1] for an easy example. The dynamics of a Hopf-zero singular system given by (1.1) may not be finitely determined with respect to topological equivalence (see [31, Definition 2.15] and [12, page 191]), and many dynamical properties such as *heteroclinic orbit breakdowns* and *Šilńikov bifurcations* cannot be detected through truncated normal form computations; see, e.g., [6, 8, 15]. However, normal form computations are still useful for the analysis of *finitely determined dynamical properties*. Finite determinacy naturally raises the notions of *n-equivalence relation* and *n-universal asymptotic unfolding* defined by Murdock [36][1] and [37].[2]

Murdock [36, 37] defines two systems as *n-equivalent* when they share all their n-jet (n-degree Taylor expansion) determined properties. The n-asymptotic unfolding is defined based on the n-equivalence relation, and it seems the most natural way of defining a versal unfolding amenable to computation and normal form analysis; see, e.g., [40, Theorem 1]. We slightly modify versal asymptotic unfolding and call it *versal asymptotic unfolding normal form*, that is, a versal asymptotic unfolding for our simplest (orbital) normal form system. This also exhibits the n-jet determined properties of all small perturbations of the original system.

In order to depict the dynamics invariant (or those not invariant) under n-equivalence relation, in subsection 5.1 we prove that bifurcations and stabilities of equilibria and small limit cycles for a generic subfamily are invariant under 2-equivalence relation, while the secondary Hopf bifurcation of an invariant torus is invariant under 3-equivalence relation. Moreover, two

[1] According to [39]: "This paper is obsolete and is replaced by the improved paper [40]. The definition of "simplified normal form" in [36] should be discarded in favor of the definition given in [37] (agreeing with [40]). Section 6.4 of [37] is based on [36] and should also be replaced by [40]. (In fact [37, Lemma 6.4.3] is sometimes incorrect in its context, although it was correct with the "simplified normal form" of [37]. This lemma is unnecessary with the method of [40])."

[2] According to [39]: "Lemma 6.4.3 is occasionally incorrect, so the method of §6.4 should be replaced by that of [40]."

more degenerate subfamilies are also discussed. The proof uses contact-equivalence relation (see [23, page 166] and [16]) and is based on computations of certain modules, called *high order term* modules. A high order term module of a map v is a module over the ring of all scalar maps (germs) and is generated by all monomial vectors like p so that $v + p$ is contact-equivalent to v. We further discuss bifurcation of a heteroclinic cycle surrounding a continuous family of invariant tori for a 2-jet normal form system; see subsection 5.2. Detection of the saddle-saddle connection simply uses the first integral of the truncated normal form and demonstrates an important advantage of our normal form representation in conservative-nonconservative terms. This heteroclinic cycle is not invariant under 2-equivalence relation, and thus it will not be pursued in our bifurcation control; see also [33, page 225].

A parametric vector field $v(\mathbf{x}, \mu)$ (for $\mu \in \mathbb{R}^p$ from a small neighborhood of the origin and $\mathbf{x} \in \mathbb{R}^3$) is called a *perturbation* or an *unfolding* for $w(\mathbf{x})$ when $v(\mathbf{x}, \mathbf{0}) = w(\mathbf{x})$. In other words, $v(\mathbf{x}, \mu) = w(\mathbf{x}) + p(\mathbf{x}, \mu)$, where $p(\mathbf{x}, \mathbf{0}) := \mathbf{0}$. The parameterized family of perturbations

$$v(\mathbf{x}, \mu) := w(\mathbf{x}) + \sum_{\{i \,|\, |\mathbf{n}_i| \leq n\}} \mu_i \mathbf{x}^{\mathbf{n}_i}, \text{ for } \mathbf{n}_i = (m_1, m_2, m_3) \in (\mathbb{N} \cup \{\mathbf{0}\})^3, |\mathbf{n}_i| = m_1 + m_2 + m_3,$$

is called the *n-jet general perturbation* for $w(\mathbf{x})$, that is, $w(\mathbf{x})$ plus all monomial perturbations of degree less than or equal to n. In order to simplify such systems, we consider a smooth locally invertible change of state variables

$$\mathbf{x} = \phi(\mathbf{y}, \mu), \text{ where the Jacobian } D_{\mathbf{y}}\phi \text{ at the origin is invertible and } \phi(\mathbf{0}, \mu) = \mathbf{0}, \tag{1.2}$$

and time rescaling

$$\tau = T(\mathbf{y}, \mu)t, \text{ with } T(\mathbf{0}, \mathbf{0}) \neq 0, \tag{1.3}$$

where t and τ denote old and new time indices. Using (1.2) and (1.3), we may transform $v(\mathbf{x}, \mu)$ into an *orbitally equivalent vector field* as

$$\tilde{v}(\mathbf{y}, \mu) := T(\mathbf{y}, \mu)(D_{\mathbf{y}}\phi)^{-1} v(\phi(\mathbf{y}, \mu), \mu). \tag{1.4}$$

Parametric vector fields will be considered modulo an equivalence, a transformation described in (1.4). The idea is to express such a family into its simplest normal form, that is, an equivalent vector field with the least possible number of monomial *terms* in its n-degree truncated (n-jet) Taylor expansion for any n. It is known that the *simplest* orbital normal form is *unique* (i.e., the normalized coefficients are uniquely determined in terms of the original system) when a *formal basis style* is chosen.

We recall that a *style* is a rule on how to choose a complement to the *removable space*, and the *removable space* is defined as the space spanned by all terms that can be simplified from the system using the equivalence (1.4). Thus, normal form style determines what terms are simplified and what terms shall remain in the normal form system. The inner product, semisimple, $\mathfrak{sl}(2)$, simplified, and formal basis styles are among the main examples; see [37, page ix] and [21, 39]. A *formal basis style* basically determines the priority of elimination between different alternative omittable terms. More precisely, in normal form computations there exist alternative monomial terms for elimination from a given vector field, and the rule on how to choose these alternative terms is called a *formal basis style*. A formal basis style is determined by an ordering on a formal basis for all vector fields (e.g., all monomial vector fields) so that those succeeding others are in priority of elimination; see [21, 22]. We assume that a formal basis style has been fixed.

From now on, we use the word *equivalence* instead of the word *orbital equivalence* given by (1.4), unless it is stated otherwise. Recall that an n-jet of a vector field refers to its n-degree Taylor expansion.

Definition 1.1. *A parametric vector field $v(\mathbf{x}, \nu)$ is called an n-*versal asymptotic unfolding normal form *for $w(\mathbf{x})$ if the following hold:*

(i) *For each small perturbation $w(\mathbf{x}) + p(\mathbf{x}, \epsilon)$ of $w(\mathbf{x})$, there exists a polynomial map $\nu(\epsilon)$ such that $\nu(\mathbf{0}) = \mathbf{0}$ and $w(\mathbf{x}) + p(\mathbf{x}, \epsilon)$ is equivalent to $v(\mathbf{x}, \nu(\epsilon))$ modulo degrees higher than or equal to $n + 1$.*

(ii) *The n-jet of $w(\mathbf{x}, \mathbf{0})$ is the n-jet of the simplest orbital normal form for $v(\mathbf{x})$.*

*We call a parametric system $v(\mathbf{x}, \nu)$ an n-*universal asymptotic unfolding normal form *for $w(\mathbf{x})$ when the following hold:*

- *$v(\mathbf{x}, \nu)$ is an n-versal asymptotic unfolding normal form for $w(\mathbf{x})$.*
- *The n-jet of $v(\mathbf{x}, \nu)$ is the n-jet of the simplest orbital normal form for the n-jet general perturbation of $w(\mathbf{x})$.*

Then we refer to the parameter ν as the universal unfolding parameter*; see also* [2, 13] *and* [31, *Definition* 2.18].

Theorem 1.2. *For any vector field $w(\mathbf{x})$ and any given natural number n, there always exists an n-universal asymptotic unfolding normal form $v(\mathbf{x}, \nu)$. Besides, for any perturbation $w(\mathbf{x}) + p(\mathbf{x}, \epsilon)$, the map $\nu(\epsilon)$ in item* (i) *is unique modulo degrees that do not affect the n-jet of $v(\mathbf{x}, \nu)$.*

Proof. As we have claimed in [21], the orbital normal form version of [21, Lemma 4.3] also holds. Hence, once a formal basis style is chosen, the simplest orbital normal form of the n-jet general perturbation of $w(\mathbf{x})$ exists, and thus its n-jet gives rise to n-universal asymptotic unfolding normal form $v(\mathbf{x}, \nu)$. The orbital normal form of a given perturbation $w(\mathbf{x}) + p(\mathbf{x}, \epsilon)$ readily gives the polynomial map $\nu(\epsilon)$ in item (i). In order to achieve item (ii), one needs to choose the formal basis style appropriately. In fact, the priority of elimination should be effectively given to parameter-independent terms rather than those terms depending on μ_i. The claim about uniqueness of $\nu(\epsilon)$ directly follows from the uniqueness of simplest orbital normal form coefficients. ■

The universal asymptotic unfolding normal form facilitates the (finitely determined) local bifurcation analysis in terms of the unfolding parameters. However, in practical engineering problems, the mathematical models are mostly involved with parameters such as control parameters; see [9]. Therefore, a practically useful approach needs to locate the bifurcations in terms of the original (control) parameters of a parametric (control) system. This has rarely been performed in the existing normal form literature; see [43] and [9, pages 99–126] for Hopf bifurcation control. Hence, a useful normal form analysis needs to compute the relations between the unfolding parameters and the parameters of the original system. This reveals the impact of control parameters on our nonlinear control system. More details will be given in section 4; the results have successfully been implemented for Hopf-zero singularities with dominant solenoidal terms in our Maple program. In order to achieve this goal, we first need to derive orbital normal forms of (1.1) and then parametric normal forms of its (small) multiple-parametric perturbations (perturbations depending on several variables).

The only existing normal form literature on hypernormalization (simplification beyond classical normal forms) of Hopf-zero singularity is that of the authors of [1, 11, 44]. All of these papers have assumed two nonzero quadratic conditions (along with other nonresonance conditions for degrees higher than two) given by

$$f_{xx}(\mathbf{0})\,(g_{yx}(\mathbf{0})+h_{zx}(\mathbf{0}))\neq 0. \tag{1.5}$$

This paper aims to complete the results on hypernormalization of Hopf-zero singularity with a nonzero quadratic part. Therefore, throughout this paper we assume a nonzero condition as

$$f_{yy}(\mathbf{0})+f_{zz}(\mathbf{0})\neq 0; \tag{1.6}$$

see also [18]. Note that the right-hand sides of inequalities (1.5) and (1.6) represent certain coefficients from the classical normal forms. Equation (1.1) is equivalent to the following normal form (in cylindrical coordinates) (see [18, Lemma 3.1]):

$$\dot{x}=\rho^2+\sum_{i=2}^{\infty}a_ix^i,\quad \dot{\rho}=\sum_{i=0}^{\infty}b_ix^{2i+1}\rho,\quad \dot{\theta}=1+\sum_{i=0}^{\infty}c_ix^{2i+1}. \tag{1.7}$$

We further assume that there exists $a_k\neq 0$ for some k. Now define

$$r:=\min\{i\mid a_i\neq 0, i\geq 1\},\quad s:=\min\{j\mid b_j\neq 0, j\geq 1\}. \tag{1.8}$$

In this paper we compute the orbital normal forms of the system (1.1), by assuming (1.6) and

$$r<s, \tag{1.9}$$

and also parametric normal forms for any of its multiple-parametric perturbations. Here we assume that $s<\infty$. The case $s=\infty$ consists of all solenoidal Hopf-zero vector fields and is discussed in [19]. The results on the other two cases ($r>s$ and $r=s$) are in progress and appear elsewhere.

Assuming that (1.6) and (1.9) hold, we prove in Theorem 3.10 that the system (1.7) is equivalent to

$$\dot{x}=2\rho^2+a_rx^{r+1}+\sum_{k=s}^{\infty}\beta_kx^{k+1},\quad \dot{\rho}=-\frac{a_r(r+1)}{2}x^r\rho+\frac{1}{2}\sum_{k=s}^{\infty}\beta_kx^k\rho,\quad \dot{\theta}=1+\sum_{k=1}^{r}\gamma_kx^k,$$

where $\beta_k=0$ for $k\equiv_{2(r+1)}-1$ and $k\equiv_{2(r+1)}s$. Finally, we prove that any multiple-parametric perturbation of the system (1.7) (for $r<s$) can be transformed into the $(s+1)$st level

parametric normal form (Theorem 4.1(i))

(1.10)

$$
\begin{aligned}
\dot{x} &= 2\rho^2 + a_{r\mathbf{0}}x^{r+1} + \sum_{-1\leq i<r+1} a_{i\mathbf{n}}x^{i+1}\mu^{\mathbf{n}} + \sum_{0\leq i<s, i\neq r} b_{i\mathbf{n}}x^{i+1}\mu^{\mathbf{n}} + \sum_{k=s}\beta_{k\mathbf{n}}x^{k+1}\mu^{\mathbf{n}},\\
\dot{\rho} &= -\frac{a_{r\mathbf{0}}(r+1)}{2}x^r\rho + \sum_{-1\leq i<r+1} a_{i\mathbf{n}}\left(\frac{i+1}{2}\right)\rho\mu^{\mathbf{n}} + \sum_{0\leq i<s, i\neq r}\frac{1}{2}b_{i\mathbf{n}}x^i\rho\mu^{\mathbf{n}} + \sum_{k=s}\frac{1}{2}\beta_{k\mathbf{n}}x^k\rho\mu^{\mathbf{n}},\\
\dot{\theta} &= 1 + \sum_{k=1}^{r}\gamma_{k\mathbf{n}}x^k\mu^{\mathbf{n}},
\end{aligned}
$$

and its $(s+1)$-universal asymptotic unfolding is given by (Theorem 4.1(ii))

$$
\begin{aligned}
\dot{x} &= 2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \sum_{1\leq i\leq r}\nu_i x^{i-1} + \sum_{i=r+1}^{N}\nu_i x^{k_i+1},\\
\dot{\rho} &= -\frac{a_r(r+1)}{2}x^r\rho + \frac{1}{2}\beta_s x^s\rho - \sum_{1\leq i\leq r}\frac{(i-1)}{2}\nu_i x^{i-2}\rho + \frac{1}{2}\sum_{i=r+1}^{N}\nu_i x^{k_i}\rho,\\
\dot{\theta} &= 1 + \sum_{i=1}^{r}(\gamma_i+\omega_i)x^i.
\end{aligned}
\tag{1.11}
$$

Here $N = r + s - \lfloor\frac{s}{2(r+1)}\rfloor$, $\mathbf{n} = (m_1, \ldots, m_p) \in (\mathbb{N}\cup\{0\})^p$, the parameters $\mu := (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p$, $\nu_i, \omega_i \in \mathbb{R}$, and $\mu^{\mathbf{n}} := \mu_1^{m_1}\ldots\mu_p^{m_p}$, and the coefficients $a_{i\mathbf{n}}, b_{i\mathbf{n}}, \beta_{i\mathbf{n}}, \gamma_{i\mathbf{n}}, \beta_s, \gamma_i \in \mathbb{R}$. The family associated with assumptions (1.6) and (1.9) are large enough so that they may appear near stagnation points associated with perturbations of incompressible fluid flows, three dimensional magnetic field lines, and well-known systems such as the Michelson system.

The rest of this paper is organized as follows. In section 2, we provide the time rescaling structure and introduce our normal form style. Section 3 presents our orbital normal form results. Parametric and universal asymptotic unfolding normal forms for $s < \infty$ are given in section 4. Normal forms in cylindrical coordinates have a phase component (i.e., $\dot{\theta}$-component in (1.7)) associated with an angle coordinate, and it is a common approach to ignore the phase component for bifurcation analysis. The obtained planar system is called an *amplitude system*. In section 5, using contact-equivalence relation, we prove that steady-state bifurcations associated with the amplitude systems for $r := 1$ and 2 are finitely determined. Then, by assuming that $r = 1$ and 2, we discuss a limited (not a complete) bifurcation analysis of the universal asymptotic unfolding normal forms. In section 6, we explain how to identify the *parameters* of a parametric system, i.e., those playing the role of asymptotic universal unfolding parameters. Roughly speaking, cognitive choices of these parameters effectively control finitely determined local dynamics of the system such as primary and secondary bifurcations of equilibria, limit cycles, and invariant tori. We have implemented our approach in Maple so that it derives the transition sets in terms of original parameters and symbolic coefficients. As far as our information is concerned, this is new in the literature of both normal form and bifurcation theory. Finally, section 7 applies our approach to an illustrating example with two imaginary uncontrollable modes. Here estimated transition sets are drawn in terms of the distinguished parameters and are supported with some numerical simulations. This demonstrates that our distinguished parameters can suitably control the local dynamics of a nonlinear Hopf-zero singular system and can be used for a possible engineering controller design.

2. Time rescaling structure and normal form style. We follow [18] and define the vector fields

$$F_k^l := (k-l+1)x^{l+1}\rho^{2(k-l)}\frac{\partial}{\partial x} - \binom{l+1}{2} x^l \rho^{2k-2l+1}\frac{\partial}{\partial \rho},$$
$$E_k^l := x^{l+1}\rho^{2(k-l)}\frac{\partial}{\partial x} + \frac{1}{2}x^l\rho^{2k-2l+1}\frac{\partial}{\partial \rho},$$
$$\Theta_k^l := -x^l\rho^{2(k-l)}\frac{\partial}{\partial \theta},$$

where the variables x, ρ, and θ represent cylindrical coordinates. Note that ρ does not take negative values, and otherwise it is singular at $\rho = 0$. Then, by [18, Theorem 2.4], any Hopf-zero normal form system (1.7) in cylindrical coordinates can be expanded in terms of F-, E-, and Θ-terms like

$$v := \Theta_0^0 + \sum a_{i,j}F_j^i + \sum b_{i,j}E_j^i + \sum c_{i,j}\Theta_j^i. \tag{2.1}$$

We denote $\mathscr{L}$ as the vector space generated by all Hopf-zero normal forms expanded with respect to F-, E-, and Θ-terms. The Lie algebra structure constants follow [18, Lemma 2.3].

We define a module structure for $\mathscr{L}$ that is instrumental for computing the effect of the near-identity time rescaling. The integral domain of formal power series (denoted by $\mathcal{R}$) generated by monomials

$$Z_n^m := x^m\rho^{2(n-m)} \tag{2.2}$$

represents the space of time rescaling generators; Z_n^m generates $t := (1 + x^m\rho^{2(n-m)})\tau$, where t and τ denote the old and new time variables. Hence, $\mathcal{R} := \operatorname{span}\{Z_n^m \mid m \leq n, m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$ acts on $\mathscr{L}$, and $\mathscr{L}$ is an $\mathcal{R}$-module.

Lemma 2.1. *The $\mathcal{R}$-module structure constants associated with time rescaling are given by*

$$Z_n^m F_k^l = \frac{k+2}{k+n+2}F_{k+n}^{l+m} + \frac{m(k+2) - n(l+1)}{k+n+2}E_{k+n}^{l+m},$$
$$Z_n^m E_k^l = E_{k+n}^{l+m},$$
$$Z_n^m \Theta_k^l = \Theta_{k+n}^{l+m}.$$

In this paper we merely apply the time rescaling space

$$\mathscr{T} := \operatorname{span}\{Z_n^m \in \mathscr{T} \mid \text{ for } m = n, \text{ or } m+1 = n\} \tag{2.3}$$

and use the formulas and identical notations from [17, 18]. Our computations and Maple program suggest that other time rescaling generators (from $\mathcal{R} \setminus \mathscr{T}$) do not simplify the system beyond what we present in this paper.

Any normal form computation requires a normal form style. Recall that for the cases with alternative terms for elimination, a formal basis style determines the priority of elimination. This is determined by an ordering on basis terms (F-, E-, Θ-terms) of $\mathscr{L}$ in a formal basis style; see [21, page 1006]. To determine our ordering, we need a grading function δ. The grading function decomposes the space $\mathscr{L}$ into δ-homogeneous vector subspaces $\mathscr{L}_i$ spanned by terms of the same grade, say grade i. The decomposition $\mathscr{L} = \sum \mathscr{L}_i$ builds a grading structure for $\mathscr{L}$, and this must make $\mathscr{L}$ a graded Lie algebra, i.e., $[\mathscr{L}_i, \mathscr{L}_j] \subseteq \mathscr{L}_{i+j}$; see [21, page 1006]. Assuming that δ is given (see, e.g., (3.3) and (4.3)), for any $v, w \in \{F_n^m, E_k^l, \Theta_q^p\}$ we define $v \prec w$ when

- Style: $\begin{cases} \delta(v) < \delta(w), \\ \delta(v) = \delta(w), \quad v = F_n^m, \text{ and } w = E_k^l \text{ or } w = \Theta_q^p, \\ \delta(v) = \delta(w), \quad v = E_k^l, \text{ and } w = \Theta_q^p. \end{cases}$

This indicates that our priority of elimination is with low grade terms over higher grades and then F-terms over E-terms. We denote $(a)_b^k := a(a+b)(a+2b)\cdots(a+(k-1)b)$ for any natural number k and real number b, and for any integer numbers m, n, p the notation

$$m \equiv_p n$$

is used when there exists an integer k such that $m - n = kp$.

3. The orbital normal forms.

Equation (1.1) can be transformed into the normal form equation (1.7) or, equivalently,

$$v^{(2)} := \Theta_0^0 + a_0 F_0^{-1} + \sum a_i F_i^i + \sum b_i E_i^i + \sum c_i \Theta_i^i, \tag{3.1}$$

where $a_i, b_i, c_i \in \mathbb{R}$; see [18, Lemma 3.1]. Without loss of generality we may assume that $a_0 = 1$; see the comments above [18, Remark 3.2]. (Note that the orbital equivalence may here include reversal of time.) Recall r and s from (1.9). Orbital normal form reduction of (3.1) is split into three cases (this is similar to the three cases of Bogdanov–Takens singularity [5, 17, 30]):

$$\text{Case I: } r < s, \qquad \text{Case II: } r > s, \qquad \text{Case III: } r = s. \tag{3.2}$$

In this paper we only deal with Case I. Recall the grading function (see [18, equation (4.3)] and [4])

$$\delta(F_k^l) = \delta(E_k^l) = r(k-l) + k, \quad \delta(\Theta_k^l) = r(k-l) + k + r \tag{3.3}$$

and the linear map

$$d^{n,N}\big(S_{n-N+1}, \dots, S_{n-r}; T_{n-N+1}, \dots, T_{n-r}\big) := \sum_{k=r}^{N-1} \big([S_{n-k}, v_k] + T_k v_{n-k}\big) \tag{3.4}$$

for any $(S_{n-N+1}, \dots, S_{n-1-r}; T_{n-N+1}, \dots, T_{n-1-r}) \in \ker d^{n-1,N-1}$, the updating vector field $v = \sum_{n=r}^{\infty} v_n$, $S_i \in \mathscr{L}_i$, and $T_i \in \mathscr{T}_i$, where $\mathscr{L}_i$, $\mathscr{T}_i$ denote the δ-homogeneous subspaces of $\mathscr{L}$ and $\mathscr{T}$. Note that the vector field v is sequentially being updated in the process of normal form computation. We skip many subtleties of the subject; see [3, 7, 17, 21, 22, 35, 38, 41] for more information.

Proposition 3.1. *Consider our formal basis style, the grading function δ in (3.3), and the linear map $d^{n,N}$ in (3.4). Then any Hopf-zero vector field v in (1.1) can be transformed into an equivalent $(N+1)$st level normal form $v^{(N+1)} = \sum w_n$ so that w_n belongs to the complement space of $\operatorname{im} d^{n,N}$. The complement space is uniquely obtained according to the normal form style.*

Proof. The proof follows from the fact that the hypotheses of [7, Theorem 6.11] hold; see also [41, Theorem A.1] and [21, Lemma 4.3]. ∎

Denote

$$\mathbb{F}_r := F_0^{-1} + a_r F_r^r.$$

We now provide some technical formulas for obtaining orbital normal forms.

Lemma 3.2 (see [19, Lemma 3.3]). *For nonnegative integers m, n, $n \geq m$, the vector field*

$$\mathcal{T}_n^m := \sum_{l=0}^{n-m-1} \frac{{a_r}^l \big((n-m-1)(r+1)-m-1\big)_{-2(r+1)}^l}{2^{l+1}\big(m+1\big)_{r+1}^{l+1}} \Theta_{n+lr}^{m+lr+l+1}$$

satisfies

$$[\mathcal{T}_n^m, \mathbb{F}_r] + \Theta_n^m = \frac{{a_r}^{n-m}\big((n-m-1)(r+1)-m-1\big)_{-2(r+1)}^{n-m}}{2^{n-m}\big(m+1\big)_{r+1}^{n-m}} \Theta_{nr-mr+n}^{nr-mr+n}.$$

Corollary 3.3. *For any nonnegative integer m, we have*

$$Z_{m+1}^m \Theta_0^0 + \left[\frac{1}{2(m+1)}\Theta_{m+1}^{m+1}, \mathbb{F}_r\right] = \frac{-a_r}{2}\Theta_{m+r+1}^{m+r+1}. \tag{3.5}$$

Besides, for each $m \neq 0$ the vector field

$$\mathcal{Z}_m^m := \frac{1}{(m+1)_1^2} F_m^m + \frac{1}{(m+2)} E_m^m \tag{3.6}$$

satisfies

$$Z_m^m \mathbb{F}_r + [\mathcal{Z}_m^m, \mathbb{F}_r] = \frac{a_r(r+1)}{m+1} F_{m+r}^{m+r}. \tag{3.7}$$

Proof. By Lemma 2.1 we have

$$\begin{aligned} Z_m^m F_0^{-1} &= \frac{2}{m+2} F_m^{m-1} + \frac{2m}{m+2} E_m^{m-1}, \\ Z_m^m F_r^r &= \frac{r+2}{m+r+2} F_{m+r}^{m+r} + \frac{m}{m+r+2} E_{m+r}^{m+r}. \end{aligned}$$

The proof is complete by a straightforward computation. ∎

We have already used the following lemma in [18, 19, 20]. Since it plays a central role in the efficient use of time rescaling for simplifying Θ-terms, we here state it as a lemma.

Lemma 3.4. *The transformation*

$$[x(t), \rho(t), \Theta(t)] = \varphi(x(t), \rho(t), \theta(t)) := [x, \rho, \theta - t], \tag{3.8}$$

where Θ and θ are the new and old phase variables, is an invertible linear change of state variables so that φ transforms away the linear part Θ_0^0 from the normal form system (2.1). In addition, φ^{-1} adds Θ_0^0 back into the normal form system.

Proof. The claim is true since the Θ-component is decoupled from the x- and ρ-components; see also the comments in [18, pages 317–318]. ∎

The first part of Corollary 3.3 implies that we can use the time rescaling associated with Z_{m+1}^m for eliminating Θ_{m+r+1}^{m+r+1}-terms. This is so when Θ_0^0 appears in the vector field. On the other hand, the second part of Corollary 3.3 implies that time rescaling terms associated with Z_m^m can be used to simplify F_{m+r}^{m+r}-terms. However, applying time rescaling Z_m^m on a vector field, whose expansion includes Θ_0^0, creates terms of the form Θ_m^m which had been simplified from the system in earlier steps using Z_{m-r}^{m-r-1}. Therefore, it is beneficial to transform Θ_0^0 away from the system when we apply time rescaling Z_m^m. Further, we transform it back to the system once we intend to apply the Z_{m+1}^m-type of time rescaling.

Corollary 3.5. *Let*

$$\mathcal{Z}_{m+1,r}^m := \frac{1}{(m+1)(m+3)} F_{m+1}^m + \frac{1}{(m+3)} E_{m+1}^m + \frac{a_r(r+1)}{2(m+1)(m+r+2)} F_{m+r+1}^{m+r+1}.$$

Then, for each $m \neq 0$, we have

$$\begin{aligned} Z_{m+1}^m \mathbb{F}_r + [\mathcal{Z}_{m+1,r}^m, \mathbb{F}_r] = \frac{{a_r}^2}{2} \Bigg(& \frac{(m-2r-1)}{(m+3)(m+r+2)} + \frac{(m-r-1)}{(m+2r+3)} \\ & - \frac{r(r+2)(m+1)}{(m+3)(m+r+2)(m+r+3)} \Bigg) F_{m+2r+1}^{m+2r+1}. \end{aligned}$$

For the case $m = 0$, the vector field

$$\mathcal{Z}_{1,r}^0 := \frac{1}{3} F_1^0 + \frac{a_r(2r^2+10r+9)}{6(r+2)(r+3)} F_{r+1}^{r+1} - \frac{a_r}{2(r+3)} E_{r+1}^{r+1}$$

satisfies

$$Z_1^0 \mathbb{F}_r + [\mathcal{Z}_{1,r}^0, \mathbb{F}_r] = -\frac{{a_r}^2(7r^2+17r+9)}{6(r+2)(2r+3)} F_{2r+1}^{2r+1} + \frac{{a_r}^2(r+1)}{2(2r+3)} E_{2r+1}^{2r+1}. \tag{3.9}$$

Proof. This is a straightforward computation along the lines of Lemma 2.1 and [18, Lemma 2.3]. ∎

In the following we use the notations $\mathcal{X}_r^k$, $\mathcal{F}_{k,r}^{-1}$, $\mathcal{T}_{k,r}^0$, and $\mathcal{E}_{k,r}^0$ along with a sequence $e_{k,m}$ which are defined by [19, equations (3.4)] and [18, equation (4.8)].

Theorem 3.6. *The $(r+1)$st level orbital normal form of (1.7) is*

$$v^{(r+1)} := \mathbb{F}_r + \sum_{k=s}^{\infty} \beta_k E_k^k + \sum_{k=0}^{r} \gamma_k \Theta_k^k,$$

and $\beta_k = 0$ *for* $k \equiv_{2(r+1)} -1$ *when* $k \geq s$.

Proof. Corollary 3.3 implies that $\Theta_m^m \in \operatorname{im} d^{m+s,r+1}$ and $F_m^m \in \operatorname{im} d^{m,r+1}$ for any $m > r$. By [18, Lemma 4.1, equation (4.10)] we have

$$[\mathcal{E}^0_{2k+1,r}, \mathbb{F}_r] = \frac{-{a_r}^{2k+2}(r+1)(2k+3)(2k+1)^{2k+1}_{-2}}{(2k)!2^{2k+1}\big((2k+2)(r+1)+1\big)} E^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r} - 2{a_r}^{2k+2} e_{2k+1,2k+2}\big(2(k+1)(r+1)+1\big) F^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r}, \tag{3.10}$$

where $e_{2k+1,2k+2}$ is nonzero. Therefore, $E^{(2k+1)(r+1)+r}_{(2k+1)(r+1)+r}$ also belongs to $\operatorname{im} d^{(2k+1)(r+1)+r,r+1}$, and the proof is complete. ■

The reader should note that the first nonzero coefficient b_j (for $j \geq 1$) from the system (1.7) may change in the $(r+1)$st level normal form computation. Consequently, the number s defined in (1.9) is changed, and it must be updated once the $(r+1)$st level normal form coefficients are computed; see also [17, Remark 2]. This is important for possible implementation of the results in a computer algebra system. We recall that

$$\ker \operatorname{ad}_{F_0^1} \circ \operatorname{ad}_{\mathbb{F}_r} = \operatorname{span}\left\{\mathcal{X}^k_r, \mathcal{F}^{-1}_{k,r}, \mathcal{E}^0_{k,r}, \mathcal{T}^0_{k,r} \,|\, k \in \mathbb{N}\right\} \tag{3.11}$$

and assume that $s < \infty$.

Define $\mathbb{X}^k_r$ by

$$\sum_{m=0}^{k}\sum_{l=0}^{2k-m+2}\binom{k}{m}\frac{{a_r}^m b_s(s)^2_2(2k-m)g_l}{2k+mr+s} F^{(r+1)(l+m)+s}_{2(k+1)+r(m+l)+s} + \sum_{m=0}^{k}\sum_{l=0}^{2k-m+2}\frac{{a_r}^{m+l} b_s\binom{k}{m}(s)^2_2(2k-m)\big(2(k-m-1)(r+1)-s\big)^l_{-2(r+1)}}{2^{l+1}\big(2k+s+(l+m)r\big)\big(m(r+1)+s\big)^{l+1}_{r+1}} E^{(r+1)(l+m)+s}_{2(k+1)+r(m+l)+s}.$$

Lemma 3.7. *For any natural number* k, *we have* $(\mathcal{X}^{k-2}_r, \mathbf{0}, \mathbb{X}^k_r; \mathbf{0}) \in \ker d^{2(k-1)(r+1)+s+r,s+1}$, *where the zeros belong to* $\mathbb{R}^{s-r-1}$ *and* $\mathbb{R}^{s+1-r}$, *respectively.*

Proof. Since

$$\sum_{m=0}^{k}\binom{k}{m}\frac{(2k-m)\big(2(k-m-1)(r+1)-s\big)^{2k-m-1}_{-2(r+1)}}{2^{2k-m-1}\big(s+m(r+1)\big)^{2k-m-1}_{r+1}} = 0, \tag{3.12}$$

we have

$$[\mathcal{X}^k_r, b_s E^s_s] + [\mathbb{X}^k_r, \mathbb{F}_r] = 0.$$

Note that the equality (3.12) is verified by Maple. ■

For ease of notation we suppress the indices of some new notations in what follows in this section. The following expression, denoted by $\mathfrak{F}$, is needed for the next lemma:

$$\sum_{m=0}^{k+1}\sum_{l=0}^{k-m}\frac{b_s {a_r}^{m+l}(s)_2^2(k+2-m)(k+2)_{-2}^{m}\big((k-2m)(r+1)-s\big)_{-2(r+1)}^{l}}{2^{m+l+1}m!\big(m(r+1)+s\big)\big(r(m+l)+k+s+2\big)\big((r+1)(m+1)+s\big)_{r+1}^{l}}E_{(m+l)r+k+s}^{(r+1)(m+l)+l}$$
$$+\sum_{m=0}^{k+1}\sum_{l=0}^{k-m}\frac{b_s {a_r}^{m}(s)_2^2(k+2-m)\big((k+2)(r+1)\big)_{-2(r+1)}^{m}g_l}{2^m m!(r+1)^m(k+mr+s+2)}F_{(m+l)r+k+s}^{(r+1)(m+l)+l}$$
$$+\sum_{m=0}^{k+1}\sum_{l=0}^{k-m}\frac{b_s {a_r}^{m+l}(s)_2^2(k+2-m)(k+2)_{-2}^{m}\big((2m-k)(r+1)+s-r\big)_{2(r+1)}^{l}}{(-1)^l 2^{m+l+1}m!(k+mr+s+2)(m(r+1)+s+1)_{r+1}^{l+1}}F_{(m+l)r+k+s}^{(r+1)(m+l)+l}.$$

Lemma 3.8. *Let k be an odd number, and let*

$$\mathbb{F}_k^{-1}:=\frac{a_r}{k+1}\mathcal{F}_k^{-1}+\frac{{a_r}^{k+2}k(r+1)(k+2)_{-2}^{k+1}}{2^{k+1}(k+1)!}\mathcal{Z}_{k(r+1)+r}^{k(r+1)+r},$$

where $\mathcal{Z}_{k(r+1)+r}^{k(r+1)+r}$ is given by (3.6). Then

$$\left(\mathbb{F}_k^{-1};\frac{{a_r}^{k+2}k(r+1)(k+2)_{-2}^{k+1}}{2^{k+1}(k+1)!}Z_{k(r+1)+r}^{k(r+1)+r}\right)\in\ker d^{k(r+1)+2r,r+1}. \tag{3.13}$$

In addition,

$$d^{k(r+1)+s+r,s+1}\left(\mathbb{F}_k^{-1},\mathbf{0},\mathfrak{F};\mathbf{0},\frac{{a_r}^{k+2}k(r+1)(k+2)_{-2}^{k+1}}{2^{k+1}(k+1)!}Z_{k(r+1)+r}^{k(r+1)+r}\right)=0.$$

Proof. The proof for the first part is complete by (3.7) and (3.6) together with [19, equation (3.4)], that is,

$$\big[\mathcal{F}_k^{-1},\mathbb{F}_r\big]=-\frac{{a_r}^{k+2}k(r+1)(k+2)_{-2}^{k+1}}{2^{k+1}(k+1)!}F_{k(r+1)+2r}^{k(r+1)+2r}.$$

By [18, Lemma 4.2] we have

$$\begin{aligned}&\big[\mathcal{F}_k^{-1},b_sE_s^s\big]+[\mathfrak{F},\mathbb{F}_r]\\&=\sum_{m=0}^{k+1}\frac{b_s{a_r}^{k+1}(s)_2^2(k+2-m)(k+2)_{-2}^{m}\big((k-2m)(r+1)-s\big)_{-2(r+1)}^{k-m}}{2^{k+1}m!\big((k+1)(r+1)+s+1\big)(m(r+1)+s)_{r+1}^{k-m-1}}E_{k(r+1)+s+r}^{k(r+1)+s+r}.\end{aligned}$$

Then the rest of the proof follows the identity

$$\begin{aligned}0&=\frac{b_s{a_r}^{k+2}k(r+1)(k+2)_{-2}^{k+1}(s+2)\big((k+1)(r+1)+s\big)}{2^{k+1}(k+1)!\big((k+1)(r+1)+s+1\big)(r+1)(k+1)}\\&\quad+\sum_{m=0}^{k+1}\frac{b_s{a_r}^{k+2}(s)_2^2(k+2-m)(k+2)_{-2}^{m}\big((k-2m)(r+1)-s\big)_{-2(r+1)}^{k-m}}{2^{k+1}m!(k+1)\big((k+1)(r+1)+s+1\big)(m(r+1)+s)_{r+1}^{k-m-1}}.\end{aligned} \tag{3.14}$$

The equality (3.14) is verified by Maple. ■

Lemma 3.9. *Let*

$$\mathbb{E}^0_{2k} := \left(\frac{a_r(r+1)}{2kr+2k+1}\right)\mathcal{E}^0_{2k,r} + {a_r}^{2k+1}e_{2k,2k}(2k(r+1)-r)\mathcal{Z}^{2k(r+1)}_{2k(r+1)}$$

for any natural number k. Then

$$\left(\mathbb{E}^0_{2k}; {a_r}^{2k+1}e_{2k,2k}\big(2k(r+1)-r\big)Z^{2k(r+1)}_{2k(r+1)}\right) \in \ker d^{2k(r+1)+r,r+1}. \tag{3.15}$$

In addition, there exists a δ-homogeneous transformation generator $\mathfrak{E} \in \mathscr{L}_{2k+s}$ *such that*

$$\begin{aligned} &d^{2k(r+1)+s,s+1}\left(\mathbb{E}^0_{2k}, \mathbf{0}, \mathfrak{E}; {a_r}^{2k+1}e_{2k,2k}(2k(r+1)-r)Z^{2k(r+1)}_{2k(r+1)}, \mathbf{0}\right) \\ &= \left(\sum_{m=0}^{2k}\frac{-b_s(k+1)\binom{k}{m}(mr+2k-s)^2_{2(s+1)}\big(2(k-m-1)(r+1)+r-s\big)^{2k-m-1}_{-2(r+1)}}{(r+1)^{-1}{a_r}^{-2k-1}2^{2k-m-1}\big(mr+2(k+1)\big)\big(m(r+1)+s+1\big)^{2k-m-1}_{r+1}}\right. \\ &\qquad \left. + b_s{a_r}^{2k+1}e_{2k,2k}\frac{(s+2)\big(2k(r+1)-r\big)^2_{r+s+1}}{\big(2k(r+1)+1\big)^2_{s+1}}\right)E^{2k(r+1)+s}_{2k(r+1)+s}. \end{aligned} \tag{3.16}$$

Proof. We recall [18, equation (4.9)] that

$$[\mathcal{E}^0_{2k,r}, \mathbb{F}_r] = {a_r}^{2k+1}e_{2k,2k}(2k(r+1)-r)F^{2k(r+1)+r}_{2k(r+1)+r}.$$

Thus, (3.7) and (3.6) imply (3.15). By [18, Lemma 4.2], for any natural number k, there exists a vector field $\mathfrak{E} \in \mathscr{L}_{2k+2}$ such that

$$\begin{aligned} &[\mathcal{E}^0_{2k,r}, b_sE^s_s] + [\mathfrak{E}, \mathbb{F}_r] \\ &= \sum_{m=0}^{2k}\frac{b_s{a_r}^{2k}(k+1)\binom{k}{m}(2k+mr-s)^2_{2s+2}\big(2(k-m-1)(r+1)+r-s\big)^{2k-m-1}_{-2(r+1)}}{2^{2k-m-1}\big(mr+2(k+1)\big)\big(m(r+1)+s+1\big)^{2k-m-1}_{r+1}}E^{2k(r+1)+s}_{2k(r+1)+s}. \end{aligned}$$

Then the rest of the proof is straightforward. ■

Theorem 3.10. *The* $(s+1)$*st level orbital normal form of* (1.7) *is*

$$v^{(s+1)} := \Theta^0_0 + F^{-1}_0 + \delta F^r_r + \sum_{k=s}^{\infty}\beta_kE^k_k + \sum_{k=1}^{r}\gamma_k\Theta^k_k. \tag{3.17}$$

Here $\delta := \operatorname{sign}(a_r)$, $\beta_k = 0$ *when* $k \equiv_{2(r+1)} -1$, *and* $k \equiv_{2(r+1)} s$ *for* $k > s$. *Furthermore, the* $(s+1)$*-jet of* $v^{(s+1)}$ *gives rise to the simplest* $(s+1)$*-jet orbital normal form.*

Proof. Using the changes of variables given above [18, Lemma 4.1], we may change the coefficient a_r to $\operatorname{sign}(a_r)$. Thus, we may assume that $a_r = \delta = \pm 1$. Lemma 3.9 (see (3.16)) concludes that terms of the form $E^{2k(r+1)+s}_{2k(r+1)+s}$ are simplified in the $(s+1)$-level. This and Theorem 3.6 imply that the normal form vector field (3.17) can be obtained in the $(s+1)$-level normal form. However, we still need to address the remaining vector fields in $\ker d^{N,r+1}$

and prove that they do not contribute to further simplification of the system in the $(s+1)$-level. In this direction, (3.11) lists the kernel terms, while Lemmas 3.7 and 3.8 prove our claim.

Since the vector space of all Θ-terms is a Lie ideal, the grading (3.3) for Θ-terms can be shifted. Therefore, in what follows we prove that the kernel term generated by Z_1^0 cannot be used for further normalizing (simplifying) E-terms (instead of Θ-terms) in the $(s+1)$-level. Equation

$$Z_{r+1}^{r+1}\mathbb{F}_r + [\mathcal{Z}_{r+1}^{r+1}, \mathbb{F}_r] = \frac{a_r(r+1)}{r+2} F_{2r+1}^{2r+1},$$

(3.10) for $k = 0$, and (3.9) imply that a linear combination (see (3.18)) of transformation generators $(\mathcal{Z}_{1,r}^0; Z_1^0)$, $\mathcal{E}_{1,r}^0$, and $(\mathcal{Z}_{r+1}^{r+1}; Z_{r+1}^{r+1})$ belongs to $\ker d^{r+s+1,r+1}$. Therefore, the linear combination of these can potentially be applied to the system for possible normalization (simplification) in the $(s+1)$-level without affecting terms of lower grades. Hence, we consider the spectral effect of either of these on the vector field $F_0^{-1} + \delta F_r^r + \beta_s E_s^s$. (Recall that Θ_0^0 can be simplified using Lemma 3.4.) This is achieved by the following formulas:

$$Z_1^0 E_s^s + \left[\frac{1}{2(s+1)} E_{s+1}^{s+1}, \mathbb{F}_r\right] = \frac{a_r r(r+2)}{2(s+1)(s+r+3)} F_{r+s+1}^{r+s+1} - \frac{a_r(s+3)}{2(r+s+3)} E_{r+s+1}^{r+s+1}$$

and

$$\begin{aligned}
&[\mathcal{Z}_{1,r}^0, E_s^s] + \left[\frac{(s)_2^2}{3(s+1)_2^2} E_{s+1}^{s+1} - \frac{1}{2(s+2)_1^2} F_{s+1}^{s+1}, \mathbb{F}_r\right] \\
&= \frac{(r^2+3r-s^2-2s+2)a_r}{2(r+s+3)\,(r+2)} E_{r+s+1}^{r+s+1} \\
&- a_r\left(\frac{(r+1)(2r^2+10r+9)}{6(r+2)_{s+1}^2} - \frac{(r)_2^2(s)_2^2}{3(s+1)_2^2(s+r+3)} - \frac{(s-r+1)}{2(s+2)_1^2}\right) F_{r+s+1}^{r+s+1},
\end{aligned}$$

while

$$\left[\mathcal{E}_{1,r}^0, b_s E_s^s\right] + \left[\frac{b_s(s-1)}{2(s+1)} E_{s+1}^{s+1}, \mathbb{F}_r\right] = -\frac{3a_r b_s r}{2(r+s+3)} E_{r+s+1}^{r+s+1} - \frac{a_r b_s r(2r+3-s)}{2(r+s+3)(s+1)} F_{r+s+1}^{r+s+1},$$

$Z_{r+1}^{r+1} E_s^s = E_{r+s+1}^{r+s+1}$, and

$$[\mathcal{Z}_{r+1}^{r+1}, E_s^s] = -\frac{r+1}{(r+2)_{s+1}^2} F_{r+s+1}^{r+s+1} + \left(\frac{s-r-1}{(r+3)} + \frac{(s)_2^2}{(r+2)_1^2(r+s+3)}\right) E_{r+s+1}^{r+s+1}.$$

Hence,

(3.18)
$$\begin{aligned}
&d^{r+s+1,s+1}\left(\mathcal{Z}_{r+1}^{r+1} + \mathcal{Z}_{1,r}^0 + \mathcal{E}_{1,r}^0, \mathbf{0}, \left(\frac{sb_s}{2(s+1)} + \frac{(s)_2^2}{3(s+1)_2^2}\right) E_{s+1}^{s+1} - \frac{1}{2(s+2)_1^2} F_{s+1}^{s+1}; Z_{r+1}^{r+1} + Z_1^0, \mathbf{0}\right) \\
&= -\frac{b_s(r+1)}{s+2} F_{r+s+1}^{r+s+1},
\end{aligned}$$

due to the equality

$$\frac{r(r-s)-(s+2)^2}{(r+2)^2_{s+1}} - \frac{r}{s+r+3} + \frac{(s+2)^2_r}{(r+2)^2_{s+1}} = 0.$$

Equation (3.18) infers that F^{r+s+1}_{r+s+1} can be simplified. However, (3.7) indicates that F^{m+r}_{m+r}-terms for each $m > 0$ have already been simplified in the $(r+1)$-level and no further simplification of E-terms is possible at this stage. ■

4. Universal asymptotic unfolding normal form. The conventional approach for local bifurcation analysis of singular parametric differential systems is first to fold the system by setting the parameters to zero and then find the normal form of the folded system. Next, the normalized system is unfolded by adding extra parameter depending terms such that the (versal) unfolded normal form system contains qualitative properties (invariant under an equivalence relation) associated with any small perturbation of the original system; see, e.g., [43]. In fact, the unfolding also accommodates all possible modeling imperfections. However, this approach has two major disadvantages. First, this approach does not provide the actual relations between the original parameters of a parametric system and the unfolding parameters. This effectively prevents its implementation to bifurcation control. Second, most singular systems do not have universal unfolding since many qualitative dynamics of systems cannot be determined by any finite jet; see [40, page 685] and [8]. The latter explains the reason why we use the notion of universal *asymptotic* unfolding normal form. Further, we use a parametric orbital normal form computation in order to compute the parameter relations. This section is devoted to treating Hopf-zero singularities (whose first few dominant terms are solenoidal) with any possible additional nonlinear degeneracies. Here the n-equivalence relation (n-jet determined) is used for introducing a universal asymptotic unfolding normal form, whose original ideas are due to [36, 37, 40] and that is amenable to finite normal form computations.

Consider the parametric differential equation

$$\dot{x} := f(x,y,z,\mu),\ \dot{y} := z + g(x,y,z,\mu),\ \dot{z} := -y + h(x,y,z,\mu), \qquad (x,y,z) \in \mathbb{R}^3,\ \mu \in \mathbb{R}^p. \tag{4.1}$$

Here f, g, and h are nonlinear formal functions in terms of (x,y,z,μ), and they are also nonlinear in terms of (x,y,z) when they are evaluated at $\mu = 0$. Remark that the results presented in this paper can be easily generalized to smooth cases using Borel–Ritt theorem [40, Theorem A.3.2]. Equation (4.1) represents a multiple-parametric perturbation of (1.1). Through a sequence of primary shifts of coordinates (shifts in y- and z-variables), we may assume that $g(0,0,0,\mu) = h(0,0,0,\mu) = 0$ for all $\mu \in \mathbb{R}^p$; see the primary and secondary shifts of coordinates in [37, page 373] and [36]. Next, it is easy to observe that the vector space spanned by all F^i_j, E^i_j, Θ^i_j is the same as the vector space spanned by all resonant vector fields; see [12, page 54]. Therefore, a normal form of (4.1) can be chosen as

$$\begin{aligned} v^{(1)} := {} & \sum c_{00\mathbf{n}}\Theta^0_0\mu^{\mathbf{n}} + \sum a_{-1,-1\mathbf{n}}F^{-1}_{-1}\mu^{\mathbf{n}} + \sum a_{-1,0\mathbf{n}}F^{-1}_0\mu^{\mathbf{n}} + \sum a_{ij\mathbf{n}}F^i_j\mu^{\mathbf{n}} \\ & + \sum b_{ij\mathbf{n}}E^i_j\mu^{\mathbf{n}} + \sum c_{ij\mathbf{n}}\Theta^i_j\mu^{\mathbf{n}}, \end{aligned} \tag{4.2}$$

where $c_{00\mathbf{0}} = 1$ and $a_{-1,-1\mathbf{0}} = 0$. It is known that the coefficients given in (4.2) are not unique and further simplification of (4.2) is possible. Using a parametric version of Lemma 3.4, we may omit $\sum c_{00\mathbf{n}}\Theta_0^0$ from the system. Now define the grading function by

$$\delta(F_k^l\mu^{\mathbf{n}}) = \delta(E_k^l\mu^{\mathbf{n}}) = \delta(\Theta_k^l\mu^{\mathbf{n}}) = k + 2|\mathbf{n}|. \tag{4.3}$$

By similar comments following [18, Lemma 3.1] and assuming that $a_{-1,0\mathbf{0}} \neq 0$, we can modify $a_{-1,0\mathbf{0}}$ into 1. Since $[F_0^0, F_0^{-1}] = -2F_0^{-1}$, we may simplify all $F_0^{-1}\mu^{\mathbf{n}}$-terms for nonzero $\mathbf{n}$. Then the formulas given in the proof of [18, Lemma 3.1] imply that the vector field can be transformed into

$$v^{(2)} := F_0^{-1} + \sum_{i\geq -1} a_{i\mathbf{n}}F_i^i\mu^{\mathbf{n}} + \sum_{i\geq 0} b_{i\mathbf{n}}E_i^i\mu^{\mathbf{n}} + \sum_{i>0} c_{i\mathbf{n}}\Theta_i^i\mu^{\mathbf{n}}, \tag{4.4}$$

denoting $a_{-1\mathbf{n}}$ for $a_{-1,-1\mathbf{n}}$. Define r, s by

$$r := \min\{i \,|\, a_{i\mathbf{0}} \neq 0\} \quad \text{and} \quad s := \min\{i \,|\, b_{i\mathbf{0}} \neq 0\}. \tag{4.5}$$

We assume that

$$r < s < \infty. \tag{4.6}$$

For further simplification, we apply a new grading structure (compare with (3.3)) generated by

$$\delta(F_k^l\mu^{\mathbf{n}}) = \delta(E_k^l\mu^{\mathbf{n}}) = r(k-l)+k+(r+1)|\mathbf{n}|,\ \delta(\Theta_k^l\mu^{\mathbf{n}}) = r(k-l)+k+s+(r+1)|\mathbf{n}|. \tag{4.7}$$

This grading facilitates the use of results from nonparametric orbital normal forms (i.e., Theorem 3.6) for parametric cases. Given

$$Z_0^0\mathbb{F}_r + \left[\frac{1}{2}F_0^0, \mathbb{F}_r\right] = \frac{a_r(r+2)}{2}F_r^r \tag{4.8}$$

and a sequence of secondary shifts in the x-variable using the equation

$$F_r^r(x+c(\mu),\rho) = F_r^r(x,\rho) + (r+1)c(\mu)F_{r-1}^{r-1}(x,\rho) + \mathcal{O}(\mu^2), \tag{4.9}$$

where $c(\mathbf{0}) = 0$, we may transform (4.4) into the $(r+1)$st level parametric normal form

$$v^{(r+1)} := F_0^{-1} + a_rF_r^r + \sum_{-1\leq i<r-1} a_{i\mathbf{n}}F_i^i\mu^{\mathbf{n}} + \sum_{0\leq i<s,} b_{i\mathbf{n}}E_i^i\mu^{\mathbf{n}} + \sum_{k=s}^{\infty}\beta_{k\mathbf{n}}E_k^k\mu^{\mathbf{n}} + \sum_{k=1}^{r}\gamma_{k\mathbf{n}}\Theta_k^k\mu^{\mathbf{n}} \tag{4.10}$$

for $a_r := \operatorname{sign}(a_{r\mathbf{0}})$. Further, denote $\beta_s := \beta_{s\mathbf{0}}$. Here $\beta_{k\mathbf{n}} = 0$ for $k \equiv_{2(r+1)} -1$ and any nonnegative integer-valued vector $\mathbf{n}$. Equation (4.8), together with

$$[E_0^0, \mathbb{F}_r] = rF_r^r, [F_0^0, E_s^s] = sE_s^s \quad \text{and} \quad [E_0^0, E_s^s] = sE_s^s,$$

imply that $E_s^s\mu^{\mathbf{n}} \in \operatorname{im} d^{s+2|\mathbf{n}|,s+1}$ when $\mathbf{n} \neq \mathbf{0}$ and $s - r \neq 0$.

For our convenience we define

$$N := r + s - \left\lfloor \frac{s}{2(r+1)} \right\rfloor$$

and a sequence of natural numbers by

$$\{k_i \,|\, i \in \mathbb{N}, i > r\} := \left\{ k \,\middle|\, k \neq s, k \geq 0, \text{ and } \frac{k+1}{2(r+1)}, \frac{k-s}{2(r+1)} \notin \mathbb{N} \right\}.$$

Now we are ready to state one of the main results of this paper.

Theorem 4.1 (universal unfolding). *Assume that condition* (4.6) *holds. Then the following hold:*

(i) *There exist an infinite sequence of formal parametric functions* $\nu_i(\mu_j)$ *and the finite sequence of formal functions* $\omega_i(\mu_j)$ *(for* $1 \leq i \leq r$*) such that* (4.1) *is equivalent to*

$$\begin{aligned} v^{(s+1)} &:= \Theta_0^0 + F_0^{-1} + a_r F_r^r + \beta_s E_s^s + \sum_{1 \leq i \leq r} \nu_i F_{i-2}^{i-2} + \sum_{i=r+1}^{N} \nu_i E_{k_i}^{k_i} \\ &\quad + \sum_{i=N+1}^{\infty} (\beta_{k_i} + \nu_i) E_{k_i}^{k_i} + \sum_{i=1}^{r} (\gamma_i + \omega_i) \Theta_i^i. \end{aligned} \tag{4.11}$$

(ii) *The differential system*

$$\begin{aligned} \dot{x} &= 2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \sum_{1 \leq i \leq r} \nu_i x^{i-1} + \sum_{i=r+1}^{N} \nu_i x^{k_i+1}, \\ \dot{\rho} &= -\frac{a_r(r+1)}{2} x^r \rho + \frac{1}{2}\beta_s x^s \rho - \sum_{1 \leq i \leq r} \frac{(i-1)}{2} \nu_i x^{i-2} \rho + \frac{1}{2} \sum_{i=r+1}^{N} \nu_i x^{k_i} \rho, \\ \dot{\theta} &= 1 + \sum_{i=1}^{r} (\gamma_i + \omega_i) x^i \end{aligned} \tag{4.12}$$

is an $(s+1)$*-universal asymptotic unfolding normal form for the differential system* (4.1).

Proof. Given the parametric normal form equation (4.10), the proof readily follows by deriving a parametric version of formulas in Theorem 3.10. The uniqueness of each polynomial map $\nu_i(\mu_j)$ and $\omega_i(\mu_j)$ follows from the uniqueness of the $(s+1)$-jet of orbital normal form. ■

5. Bifurcation analysis and finite determinacy. In this section, we prove that the 2- and 3-equivalence relations are compatible with the contact-equivalence relation for the cases of $r := 1$ and 2, respectively. Therefore, the steady-state bifurcations and stabilities of equilibria associated with the amplitude system are 2- and 3-determined, respectively. When $r := 1$, the secondary Hopf bifurcation is 3- and 5-determined for $s := 2$ and 3, respectively. The proofs follow a systematic approach and well-established theory; see, e.g., [23]. We analyze the local bifurcations of equilibria, limit cycles, and secondary Hopf bifurcations of invariant tori of the appropriate finite jet normal form system. For these cases, the hypernormalization up to the $(s+1)$-level is necessary and sufficient for analysis and control of these primary and secondary bifurcations. Moreover, the coefficients of $(s+1)$-level normal form are interestingly sufficient to directly determine the existence and stability type of secondary Hopf bifurcations to invariant tori. The latter indeed only depends on the appropriate unfolding parameter along with β_2 and β_4 for $s := 2$ and 3, respectively; see subsections 5.2.2 and 5.2.3.

5.1. Finite determinacy. Finite determinacy or jet sufficiency is one of the most important challenges in local bifurcation analysis of normal form singular systems. The question is whether a finite jet, say k, of the normal form system has the same qualitative behavior as the original system and no information would be lost by only considering a k-jet of its normal form. In the affirmative case, we say that the system is k-determined. In this section we are concerned about the roots of an amplitude normal form system. When k-jet sufficiency is proved, the root bifurcations for a k-jet of normal form correspond to bifurcations of the original system. We use *contact-equivalence* in this section, that is, the most natural equivalence relation preserving local roots of smooth maps (germs); see [16] and [23, page 166].

Finitely determined results for Hopf-zero singularities of codimension two have been reported in the literature using C^0- and weak C^0-equivalences; see, e.g., [14, 42]. We recall that the three dimensional Hopf-zero singularity may demonstrate complex dynamical behaviors such as births/deaths of *invariant tori*, *phase locking*, *chaos*, *strange attractors*, *heteroclinic orbit breakdowns*, and *Šilńikov bifurcations* which may not be detected by singularity theory and/or normal form methods; see [25, 32, 33, 34]. In fact, we merely address the bifurcation problem of equilibria, limit cycles, and secondary Hopf bifurcations of invariant tori; see also [34].

We first claim that the normal form computation of vector fields and application of results from singularity theory are compatible. It is well known that for any smooth differential system (1.1) there always exist C^∞-smooth changes of coordinates to transform (1.1) into a smooth differential system (1.7) modulo flat parts; see, e.g., [40, Theorem 1]. The transformed equation is further reduced by ignoring the phase component to obtain the *amplitude system*. Since further smooth changes of coordinates and time rescaling transform a smooth germ to other contact-equivalent germs, we may instead work with a reduced system obtained from a universal asymptotic unfolding normal form.

Throughout this subsection we follow the notations, terminologies, and results of singularity theory from [23, Chapter XIV]. The bifurcations of limit cycles and equilibria are in one-to-one correspondence with those of equilibria for the amplitude system. Hence, we define the map $F = (F_1, F_2)$ by

$$F := \left(\nu_1 + \nu_2 x + 2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \text{h.o.t.}, \frac{\nu_2 \rho}{2} - a_r x^r \rho + \frac{\beta_s x^s \rho}{2} + \text{h.o.t.}\right). \tag{5.1}$$

Here h.o.t. stands for smooth maps whose $(s+1)$-jet is zero. The distinguished parameter λ is chosen as ν_2 for $r := 1$ and ν_1 for $r := 2$, and we remove the remaining parameter by setting it to zero.

5.1.1. The case $r := 1$. We let $\lambda := \nu_2$ and $\nu_1 := 0$. The symmetry group Γ is the trivial (identity) group, and thus it is removed in our notations. Since $r := 1$,

$$F(x,\rho,\lambda) := \left(\lambda x + 2\rho^2 + a_1x^2 + \beta_s x^{s+1} + \text{h.o.t.}, \frac{\lambda\rho}{2} - a_1 x\rho + \frac{\beta_s x^s \rho}{2} + \text{h.o.t.}\right). \tag{5.2}$$

Now we recall some notations from [23, Chapter XIV]. The local ring of all *smooth scalar germs* in (x,ρ,λ)-variables is denoted by $\mathscr{E}_{x,\rho,\lambda}$. Next we define $\mathcal{M}$ as the unique maximal ideal of $\mathscr{E}_{x,\rho,\lambda}$, i.e., $\mathcal{M} := \langle x,\rho,\lambda\rangle_{\mathscr{E}_{x,\rho,\lambda}}$. Further, denote $\overrightarrow{\mathcal{M}}$ for a module over $\mathscr{E}_{x,\rho,\lambda}$ defined by

$$\overrightarrow{\mathcal{M}} := \left\langle \begin{pmatrix} x\\0\end{pmatrix}, \begin{pmatrix} 0\\x\end{pmatrix}, \begin{pmatrix} \rho\\0\end{pmatrix}, \begin{pmatrix} 0\\\rho\end{pmatrix}, \begin{pmatrix} \lambda\\0\end{pmatrix}, \begin{pmatrix} 0\\\lambda\end{pmatrix} \right\rangle_{\mathscr{E}_{x,\rho,\lambda}}. \tag{5.3}$$

Now one can imagine notations like $\overrightarrow{\mathcal{M}}^3$ and $\mathcal{M}^3$. The module $\overrightarrow{\mathcal{M}}^3$ includes all vector fields whose 2-jet is zero; in particular, $\overrightarrow{\mathcal{M}}^3$ includes all flat vector fields. Further, $\overrightarrow{\mathcal{M}}^3 = \mathcal{M}^3\binom{1}{0} + \mathcal{M}^3\binom{0}{1}$.

Lemma 5.1. *The map (germ) $F(x,\rho,\lambda)$ is contact equivalent to $F+p$ for any $p \in \overrightarrow{\mathcal{M}}^3$.*

Proof. We follow [23, Definition 7.1, Proposition 1.4, and Theorems 7.2 and 7.4] and instead prove that $\overrightarrow{\mathcal{M}}^3 \subseteq \mathcal{K}_s(F)$. The $\mathscr{E}_{x,\rho,\lambda}$-module $\mathcal{K}_s(F)$ is the intrinsic part of the module generated by

$$\mathcal{M}^2\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix}, \mathcal{M}^2\begin{pmatrix} F_{1\rho}\\F_{2\rho}\end{pmatrix}, \mathcal{M}\begin{pmatrix} F_1\\0\end{pmatrix}, \mathcal{M}\begin{pmatrix} F_2\\0\end{pmatrix}, \mathcal{M}\begin{pmatrix} 0\\F_1\end{pmatrix}, \mathcal{M}\begin{pmatrix} 0\\F_2\end{pmatrix}. \tag{5.4}$$

We choose $a_1 := 1$ to simplify the formulas. For any $\mathscr{E}_{x,\rho,\lambda}$-modules J and $\mathcal{K}_s$, the Nakayama lemma [23, Facts 2.4iii, page 251] implies that $J \subseteq \mathcal{K}_s$ if and only if $J \subseteq \mathcal{K}_s + \mathcal{M}J$. Given $J := \overrightarrow{\mathcal{M}}^3$ and $\overrightarrow{\mathcal{M}}^4 = \mathcal{M}J$, we denote $\cong$ for the equations modulo $\overrightarrow{\mathcal{M}}^4$. Since

$$\rho\begin{pmatrix} F_1\\0\end{pmatrix} - 2x\begin{pmatrix} F_2\\0\end{pmatrix} \cong \begin{pmatrix} 3x^2\rho + 2\rho^3\\0\end{pmatrix}, \; x\rho\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix} - 2x\begin{pmatrix} F_2\\0\end{pmatrix} \cong \begin{pmatrix} 4x^2\rho\\-x\rho^2\end{pmatrix},$$
$$\rho x\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix} - \rho\begin{pmatrix} F_1\\0\end{pmatrix} \cong \begin{pmatrix} x^2\rho - 2\rho^3\\-x\rho^2\end{pmatrix},$$

we have $\binom{x^2\rho}{0}, \binom{\rho^3}{0}, \binom{0}{x\rho^2} \in \mathcal{K}_s + \mathcal{M}^4$. Further,

$$x\begin{pmatrix} F_2\\0\end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}x\lambda\rho - x^2\rho\\0\end{pmatrix}, \lambda\begin{pmatrix} F_2\\0\end{pmatrix} \cong \begin{pmatrix} \frac{1}{2}\lambda^2\rho - x\lambda\rho\\0\end{pmatrix}, \rho\begin{pmatrix} F_{1\rho}\\F_{2\rho}\end{pmatrix} - \begin{pmatrix} 0\\F_2\end{pmatrix} \cong \begin{pmatrix} 4\rho^2\\0\end{pmatrix}$$

infer that $\binom{x\lambda\rho}{0}, \binom{\lambda^2\rho}{0}, \binom{\rho^2\lambda}{0}, \binom{\rho^2 x}{0} \in \mathcal{K}_s$ modulo $\mathcal{M}^4$. Next, by

$$\rho^2\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix}, \lambda\begin{pmatrix} 0\\F_2\end{pmatrix}, \begin{pmatrix} 0\\F_1\end{pmatrix} + x\begin{pmatrix} F_{1\rho}\\F_{2\rho}\end{pmatrix} \cong \begin{pmatrix} 4x\rho\\2\rho^2 + \frac{3}{2}x\lambda\end{pmatrix}, x\begin{pmatrix} 0\\F_2\end{pmatrix}, x\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix} - 2\begin{pmatrix} F_1\\0\end{pmatrix} \cong \begin{pmatrix} -x\lambda\\x\rho\end{pmatrix}, \rho\begin{pmatrix} 0\\F_2\end{pmatrix},$$

we may imply the membership of $\binom{0}{\rho^3}, \binom{0}{x\lambda\rho}, \binom{0}{\lambda^2\rho}, \binom{0}{x^2\rho}, \binom{\lambda x^2}{0}, \binom{0}{\lambda\rho^2}$ in $\mathcal{K}_s + \mathcal{M}^4$. Finally,

$$2\lambda\begin{pmatrix} F_1\\0\end{pmatrix} - \lambda x\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix} \cong \begin{pmatrix} x\lambda^2 + 4\lambda\rho^2\\x\lambda\rho\end{pmatrix}, \lambda^2\begin{pmatrix} F_{1x}\\F_{2x}\end{pmatrix}, \begin{pmatrix} 0\\F_1\end{pmatrix} + x\begin{pmatrix} F_{1\rho}\\F_{2\rho}\end{pmatrix} \cong \begin{pmatrix} 4x\rho\\2\rho^2 + \frac{3}{2}x\lambda\end{pmatrix}, \lambda^2\begin{pmatrix} F_{1\rho}\\F_{2\rho}\end{pmatrix},$$

and $x\binom{0}{F_1}$ conclude the memberships of $\binom{x\lambda^2}{0}$, $\binom{\lambda^3}{0}$, $\binom{0}{\lambda x^2}$, $\binom{0}{x\lambda^2}$, $\binom{0}{\lambda^3}$, and $\binom{0}{x^3}$. Now we have proved that all generators of $\overrightarrow{\mathcal{M}}^3$ belong to $\mathcal{K}_s(F)$ modulo $\overrightarrow{\mathcal{M}}^4$. This completes the proof by the Nakayama lemma. ■

Remark 5.2. Lemma 5.1 implies that any perturbation $F+p$ for $p \in \overrightarrow{\mathcal{M}}^3$, including the flat perturbations, of the map F in (5.2) is contact equivalent to F. Thus, the number of equilibria and limit cycles for the original three dimensional system is 2-determined and is invariant under flat perturbations. This plays an important role in the application of normal forms to bifurcation theory due to the Borel lemma; that is, any formal normal form is a smooth normal form modulo a flat function. We recall that subordinate Šil'nikov homoclinic intersections in Hopf-zero singularities are not invariant under flat perturbations, and hence their bifurcations cannot be treated by crude uses of normal form methods; see [8].

Corollary 5.3. *The* 2*-jet of the map* F *in* (5.1) *is a universal unfolding for* $r := 1$ *with respect to* $\mathbb{Z}_2$*-contact equivalence relation, where* $\mathbb{Z}_2 := \{\sigma, I\}$ *and* $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ *is given by* $\sigma(x,\rho) := (x, -\rho)$.

Proof. The proof follows the equivariant universal unfolding theorem [23, Theorem 2.1 and equation (2.7), page 211], the proof of Lemma 5.1, and $T(F) = \overrightarrow{\mathcal{M}}(\mathbb{Z}_2)$. ■

5.1.2. The case $r := 2$. In the following lemma we prove that the equilibria of an amplitude normal form system associated with (4.12) for $r := 2$ is 3-determined within the family of $\mathbb{Z}_2$-equivariant maps. Here $\mathbb{Z}_2$ is generated by $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$, $\sigma(x,\rho) := (x,-\rho)$. Thereby,

$$F(x,\rho,\lambda) := \left(\lambda + 2\rho^2 + a_2x^3 + \beta_s x^{s+1} + \text{h.o.t.}, -\frac{3}{2}a_2x^2\rho + \frac{\beta_s x^s\rho}{2} + \text{h.o.t.}\right). \tag{5.5}$$

Lemma 5.4. *The map* F *is* $\mathbb{Z}_2$*-contact equivalent to* $F+p$ *for any* $p \in \overrightarrow{\mathcal{M}}^4(\mathbb{Z}_2)$.

Proof. The proof closely follows the proof of Lemma 5.1, the $\mathbb{Z}_2$-equivariant and -invariant structures, Nakayama's lemma, and the following equalities modulo $\overrightarrow{\mathcal{M}}^5(\mathbb{Z}_2)$:

$$p\begin{pmatrix} F_1 \\ 0 \end{pmatrix} \cong \begin{pmatrix} p\lambda \\ 0 \end{pmatrix}, p\begin{pmatrix} 0 \\ F_1 \end{pmatrix} \cong \begin{pmatrix} 0 \\ p\lambda \end{pmatrix}, p\begin{pmatrix} F_{1\rho} \\ F_{2\rho} \end{pmatrix} \cong \begin{pmatrix} 4p\rho \\ 0 \end{pmatrix},$$

for any monomial p with $\deg(p) = 3$, and

$$x\begin{pmatrix} 0 \\ F_2 \end{pmatrix} \cong \begin{pmatrix} 0 \\ -\frac{3}{2}x^3\rho \end{pmatrix}, x^2\begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix} \cong \begin{pmatrix} 3x^4 \\ -3x^3\rho \end{pmatrix}, \rho^2\begin{pmatrix} F_{1x} \\ F_{2x} \end{pmatrix} \cong \begin{pmatrix} 3x^2\rho^2 \\ -3x\rho^3 \end{pmatrix}.$$ ■

5.2. Bifurcation analysis. In this section we discuss the local bifurcations of equilibria, limit cycles, a torus, and a heteroclinic cycle. Here three of the most generic families (($r := 1, s := 1$), ($r := 1, s := 2$), and ($r := 2, s := 3$)) are considered; see [25, 32, 33, 34] and the references therein for the existing literature.

5.2.1. The case $r := 1$. The amplitude system associated with the 2-jet universal asymptotic unfolding normal form v is given by

$$\dot{x} = \nu_1 + 2\rho^2 + \nu_2 x + a_1x^2, \quad \dot{\rho} = \frac{1}{2}\nu_2\rho - a_1x\rho. \tag{5.6}$$

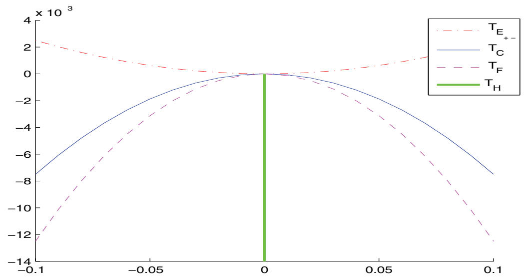


Figure 1. *Bifurcation varieties (transition sets) for the system* (5.6) *with* $a_1 = 1$*: The vertical and horizontal axes stand for* ν_1 *and* ν_2*, respectively.*

Note that the x-axis is always an invariant line. The associated equilibria follow:

$$E_\pm : (x_{E_\pm}, \rho_{E_\pm}) = \left(\frac{-\nu_2 \pm \sqrt{{\nu_2}^2 - 4a_1\nu_1}}{2a_1}, 0\right) \text{ and } C : (x_C, \rho_C) = \left(\frac{1}{2a_1}\nu_2, \sqrt{-\frac{1}{2}\nu_1 - \frac{3}{8a_1}\nu_2^2}\right).$$

The points $E_\pm$ represent equilibria, while C represents a limit cycle for the three dimensional system. The transition varieties associated with $E_\pm$ and C (depicted in Figure 1) are governed by

$$T_{E_\pm} := \left\{(\nu_1, \nu_2) \mid \nu_1 = \frac{1}{4a_1}{\nu_2}^2\right\} \quad \text{and} \quad T_C := \left\{(\nu_1, \nu_2) \mid \nu_1 = -\frac{3}{4a_1}{\nu_2}^2\right\}.$$

Each of these transition varieties $T_{E_\pm}$ and T_C is associated with a saddle-node bifurcation.

The eigenvalues of matrices $Dv(E_\pm)$ are given by $\pm\sqrt{{\nu_2}^2 - 4a_1\nu_1}$ and $\nu_2 \mp \frac{1}{2}\sqrt{{\nu_2}^2 - 4a_1\nu_1}$. When $(\nu_2 < 0)$ or $(\nu_2 > 0$ and $4a_1\nu_1 < -3{\nu_2}^2)$ hold, the equilibrium E_+ is a saddle point. For $\nu_2 > 0$, $4a_1\nu_1 > -3{\nu_2}^2$, E_+ is a source. On the other hand, the conditions $\nu_2 > 0$ or $\nu_2 < 0$ and $4a_1\nu_1 < -3{\nu_2}^2$ imply that E_- is a saddle point. However, the conditions $\nu_2 < 0$ and $4a_1\nu_1 > -3{\nu_2}^2$ conclude that E_- is a sink.

The eigenvalues of $Dv(C)$ are $\lambda_\pm = \nu_2 \pm \frac{1}{2}\sqrt{8a_1\nu_1 + 10{\nu_2}^2}$. Thus, we define

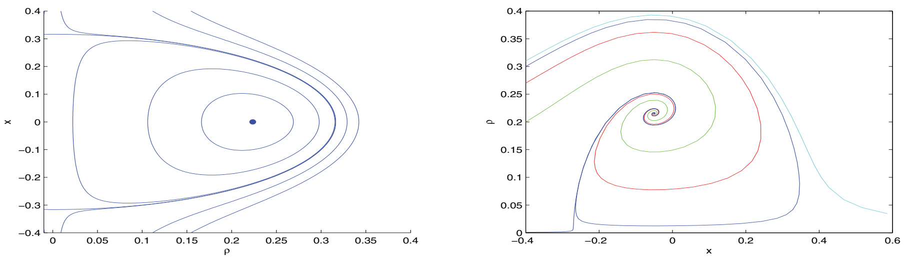
$$T_F := \left\{(\nu_1, \nu_2) \mid \nu_1 = -\frac{5}{4a_1}{\nu_2}^2\right\}. \tag{5.7}$$

For $4a_1\nu_1 < -5{\nu_2}^2$, C is a stable/unstable focus point for negative/positive values for ν_2. If $-5{\nu_2}^2 < 4a_1\nu_1 < -3{\nu_2}^2$ holds, for $\nu_2 > 0$ the point C is a source, while $\nu_2 < 0$ concludes that C is a sink. A pair of pure imaginary eigenvalues occurs at

$$T_H := \{(\nu_1, \nu_2) \mid \nu_2 = 0, a_1\nu_1 < 0\}.$$

The system (5.6) represents a nonlinear center when the parameters cross the variety T_H. In fact, the system has a first integral given by

$$I(x, \rho) := \nu_1\rho^2 + \rho^4 + a_1x^2\rho^2.$$



(a) Heteroclinic cycle for $\nu_2 := 0$. (b) Orbits for $\nu_2 := -0.1$.

Figure 2. *Saddle-saddle connecting cycle surrounding the center point at $\nu_2 := 0$ and its bifurcation to the stable focus point at $\nu_2 := -0.1$ for the two dimensional system (5.6), $a_1 := 1$, and $\nu_1 := -0.1$.*

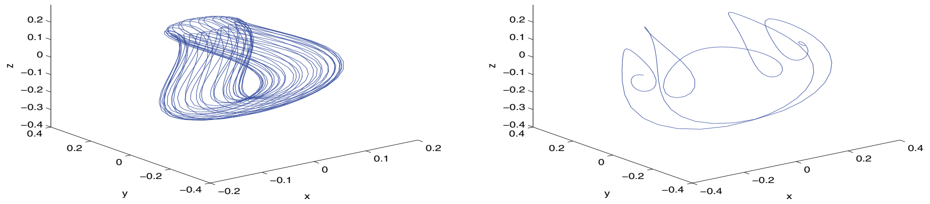


Figure 3. *An attracting torus and an estimated saddle-saddle connecting orbit for system (5.6) (along with $\dot{\theta} := 1$ for the three dimensional system) at $\nu_2 := 0$, $a_1 := 1$, and $\nu_1 := -0.1$. Initial values are chosen as $(x, \rho, \theta) := (0.1, 0.2, 0)$ and $(x, \rho, \theta) := (-0.28, 0, 0.02)$, respectively.*

This first integral is readily derived given our normal form representation. The saddle equilibria E_+ and E_- belong to saddle-saddle connection cycle surrounding a center in the two dimensional (x, ρ)-system; see Figure 2. The saddle-saddle connection is translated in the three dimensional system as an orbit on a 2-sphere surrounding a continuous family of invariant tori around a limit cycle; see Figure 3. As ν_2 crosses to positive values, the saddle-saddle heteroclinic cycle breaks; see Figure 2(b). However, this heteroclinic cycle is not invariant under 2-equivalence relations and shall not be pursued in our bifurcation control; see also [33, page 225].

5.2.2. The case $r := 1$ and $s := 2$. Generically, the family of systems for $r := 1$ at the equilibrium C undergoes a secondary Hopf bifurcation giving birth to an (attracting) invariant torus for ($\beta_2 < 0$ in) the three dimensional system. Furthermore, this secondary bifurcation is 3-determined for $\beta_2 \neq 0$. Consider $a_1 := 1$ in the 3-jet system given by

$$\dot{x} = \nu_1 + 2\rho^2 + \nu_2 x + x^2 + \beta_2 x^3 + \nu_3 x^2, \quad \dot{\rho} = \frac{1}{2}\nu_2\rho - x\rho + \frac{\beta_2}{2}x^2\rho + \nu_3 x\rho, \tag{5.8}$$

where $\beta_2 \neq 0$. The qualitative dynamics of equilibria for (5.6) and (5.8) are equivalent due to Lemma 5.1. Thus, we only consider the bifurcation point C when β_2 and ν_3 are also taken

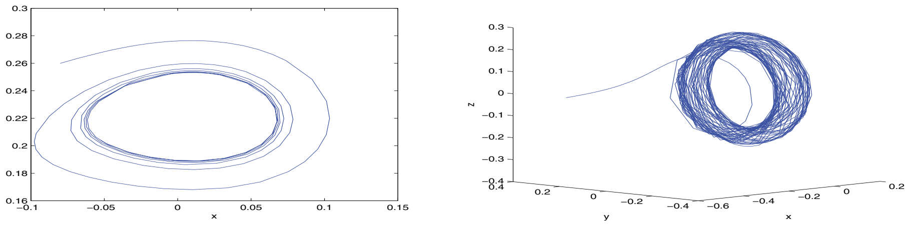


Figure 4. *Orbits approaching a stable focus for 2D-system (5.8) and an attracting invariant torus for 3D-system (equation (5.8) along with $\dot{\theta} := 1$) at $(\nu_1, \nu_2, \nu_3) := (-0.1, 0.02, 0.1)$, and $\beta_2 := -10$.*

into consideration, whose Taylor approximation is given by

$$(x_C, \rho_C) := \left(\frac{1}{2}\nu_2 + \frac{\beta_2}{8}{\nu_2}^2 + \frac{1}{4}\nu_2\nu_3, \frac{\sqrt{-2{\beta_2}^2\nu_1 - \frac{3}{2}{\beta_2}^2{\nu_2}^2 + \frac{3}{4}{\beta_2}^2{\nu_2}^2\nu_3 + \frac{3}{2}\beta_2{\nu_3}^2\nu_2}}{2\,\mathrm{sign}(\beta_2)\,\beta_2}\right).$$

This has a Hopf singularity at $\nu_2 := 0$, and hence

$$T_H := \{(\nu_1, 0, \nu_3)\} \quad \text{and} \quad (x_{C_H}, \rho_{C_H}) := \left(0, \frac{\sqrt{2}}{2}\sqrt{-\nu_1}\right). \tag{5.9}$$

By using our Maple program [21, 22], the 3-jet parametric normal form of the system (5.8) in polar coordinates (ϱ, ϑ) gives rise to

$$\dot{\varrho} := \left(2\beta_2{\nu_2}^2 + 3\nu_3\nu_2 + 4\nu_2\right)\varrho + 32\beta_2\varrho^3.$$

For $\beta_2 > 0$ and small values of $\nu_2 < 0$, the system experiences an unstable limit cycle, while it has a stable limit cycle when $\beta_2 < 0$ and $\nu_2 > 0$. This is translated to a bifurcation of an attracting invariant torus (for $\beta_2 < 0$) in the original three dimensional system; see Figure 4, where the initial values are chosen as $(x, \rho, \theta) := (-0.57, 0.2, 0)$. It can be seen that ν_1 and ν_2 are sufficient to analyze, detect, and locate the bifurcation of an invariant torus for (5.8). Thus, ν_3 is omitted in a bifurcation controller design.

5.2.3. The case $r := 1$ and $s := 3$. Now we investigate a subordinate bifurcation leading to an invariant torus for a degenerate case where $a_1 := 1$, $\beta_4 \neq 0$, $r := 1$, and $s := 3$, i.e., $\beta_2 := 0$, $\beta_3 \neq 0$. This bifurcation is 5-determined, and thus we consider

$$\begin{aligned} \dot{x} &= \nu_1 + \nu_2 x + \nu_3 x^2 + x^2 + 2\rho^2 + \beta_3 x^4 + \beta_4 x^5, \\ \dot{\rho} &= -x\rho + \frac{1}{2}\nu_2\rho + \frac{1}{2}\nu_3 x\rho + \frac{1}{2}\beta_3 x^3\rho + \frac{1}{2}\beta_4 x^4\rho. \end{aligned} \tag{5.10}$$

The Hopf singularity occurs at $\nu_2 := 0$, say C_H, given by (5.9) for $\nu_1 < 0$. Let $w := 2\sqrt{-2\nu_1}$, and treat the bifurcation parameters (ν_2, ν_3) as $||(\nu_2, \nu_3)|| = o(|\nu_1|^2)$. By using our Maple

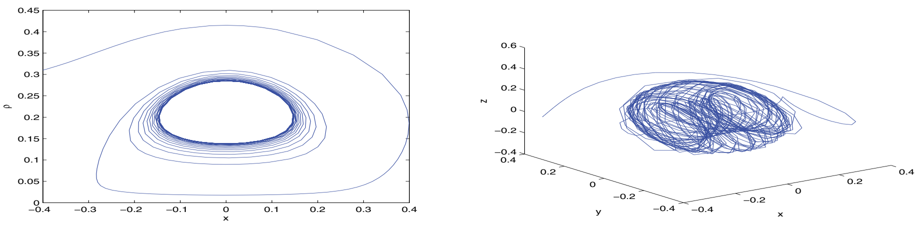


Figure 5. *Orbits approaching a stable limit cycle for two dimensional system* (5.10) *and an attracting invariant torus for the three dimensional system* ((5.10) *along with* $\dot\theta := 1$) *at* $\nu_1 := -0.1$, $\nu_2 := 0.001$, $\nu_3 := 0.001$, $\beta_2 := 0$, $\beta_3 := -1$, *and* $\beta_4 := -10$.

program for parametric normal forms of Hopf singularity in polar coordinates (ϱ, ϑ),

$$
\begin{aligned}
\dot\varrho := & \left(4\nu_2 + 3\nu_2\nu_3 + \frac{17}{16}\nu_2{\nu_3}^2 + \frac{33}{32}\beta_3{\nu_2}^3\right)\varrho \\
& + \left(99\frac{\nu_2}{w^2} + 60\beta_3\nu_2 + 72\beta_4{\nu_2}^2 + \frac{249}{4}\beta_3\nu_2\nu_3 + 1002\frac{\nu_2\nu_3}{w^2}\right)\varrho^3 \\
& + \left(384\beta_4 + 502{\beta_3}^2\nu_2 + 288\nu_3\beta_4 + \frac{683617}{12}\frac{\nu_2}{w^4} + \frac{645385}{24}\frac{\beta_3\nu_2}{w^2}\right)\varrho^5.
\end{aligned}
$$

Therefore, we have no limit cycle when $\beta_4\nu_2 > 0$, while we have one stable limit cycle for $\beta_4 < 0$ and $\nu_2 > 0$. An unstable limit cycle occurs for $\nu_2 < 0$ and $\beta_4 > 0$. Bifurcation of a stable limit cycle here means a secondary bifurcation of an attracting invariant torus for the original Hopf-zero system; see Figure 5.

5.2.4. The case $r := 2$ and $s := 3$. This case constitutes a degenerate Hopf-hysteresis singularity; see [33, 34] for a comprehensive literature. For the case of $r := 2$, we consider

$$
\begin{aligned}
\dot x &= \nu_1 + \nu_2 x + \nu_3 x + 2\rho^2 + a_2 x^3 + \beta_3 x^4, \\
\dot\rho &= -\frac{1}{2}\nu_2\rho + \frac{1}{2}\nu_3\rho - \frac{3a_2}{2}x^2\rho + \frac{\beta_3}{2}x^3\rho,
\end{aligned}
\tag{5.11}
$$

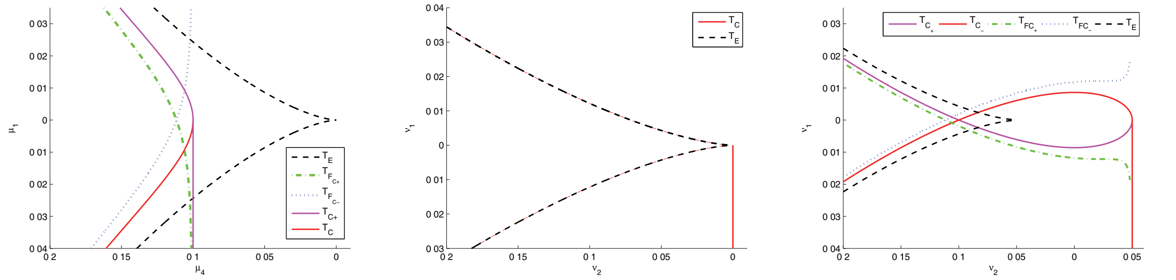
where $a_2 = \pm 1$. We assume that $a_2 := 1$, and by Lemma 5.4 for bifurcations of equilibria and limit cycles, we merely consider its 3-jet. There are two categories of equilibria for the system (5.11). The first category is like E_i given by $(x_i, 0)$ for $i = 1, 2, 3$. In fact, two of these equilibria, say E_2 and E_3, are born in a saddle-node bifurcation when the parameters cross the variety

$$
T_E : \qquad \left\{(\nu_1, \nu_2, \nu_3) : 27{\nu_1}^2 + 4(\nu_2 + \nu_3)^3 = 0\right\}.
$$

The second category follows:

$$
C_\pm : \qquad (x_{C_\pm}, \rho_{C_\pm}) = \left(\pm\frac{1}{3}\sqrt{3\nu_3 - 3\nu_2}, \frac{1}{6}\sqrt{-18\nu_1 \mp 4(\nu_2 + 2\nu_3)\sqrt{3\nu_3 - 3\nu_2}}\right).
$$

The equilibria $C_\pm$ represent two limit cycles for the three dimensional system (the normal form equation (5.11) along with the phase component). The eigenvalues of C_+ and C_- are



(a) $\nu_3 := -0.1$ (b) $\nu_3 := 0$ (c) $\nu_3 := 0.05$

Figure 6. *Transition sets T_E, T_C, and $T_{FC_\pm}$ for $r := 2$, $s := 3$, (5.11). Transition sets of T_{FC_+}/T_{FC_-} (and also T_{C_+}/T_{C_-}) coincide with the upper/lower branch of the cusp T_E for $\nu_3 := 0$. There are three equilibria E_i for $i := 1, 2, 3$ inside the cusp variety T_E and one equilibrium E_1 outside the cusp. The points $C_\pm$ exist/do not exist for parameters below/above the varieties $T_{C_\pm}$, respectively. For parameters above/below the variety T_{FC_+}, C_+ is a node/focus, while C_- is always an unstable node.*

given by

$$\nu_3 \pm \frac{\sqrt{3}}{3}\sqrt{11{\nu_3}^2 + 6\nu_1\sqrt{3\nu_3 - 3\nu_2} - 4\nu_2(\nu_2 + \nu_3)}$$

and

$$\nu_3 \pm \frac{\sqrt{3}}{3}\sqrt{11{\nu_3}^2 - 6\nu_1\sqrt{3\nu_3 - 3\nu_2} - 4\nu_2(\nu_2 + \nu_3)},$$

respectively. Thus, two equilibria are born in a saddle-node bifurcation when the parameter crosses

$$T_{C_\pm} := \left\{ \left(\mp\frac{2\sqrt{3}}{9}(\nu_2 + 2\nu_3)\sqrt{\nu_3 - \nu_2}, \nu_2, \nu_3 \right) : \nu_2 \leq \nu_3 \right\} \cup \{(\nu_1, \nu_2, \nu_2) : \nu_1 \leq 0\}.$$

The equilibria $C_\pm$ are both unstable when $\nu_3 > 0$. For $\nu_3 < 0$, the equilibrium C_+ is always stable, while C_- is unstable. Define

$$T_{FC_\pm} := \left\{(\nu_1, \nu_2, \nu_3) : 11{\nu_3}^2 - 4{\nu_2}^2 - 4\nu_2\nu_3 \pm 6\nu_1\sqrt{3\nu_3 - 3\nu_2} = 0, \nu_3 \neq 0\right\}.$$

The equilibrium C_+ changes from the node type of equilibrium to a focus point when the parameters cross T_{FC_+}; see Figure 6 and [25, Figure 6] for a similar approach. Since the transition variety T_{FC_-} is always above T_{C_-}, C_- is always an unstable node. When the parameter $\nu_3 := 0$, C_+ has a Hopf singularity, while C_- is a saddle point. Therefore, the system may generically experience a secondary bifurcation at C_+. This is beyond the scope of this paper and will not be discussed here.

6. Bifurcation control and universal asymptotic unfolding. Bifurcation control refers to designing a controller for a nonlinear system so that its dynamics gains a desirable behavior; see [9]. This has many important engineering applications and has attracted many researchers. Normal form theory is a powerful tool for local bifurcation control, and recently it has been efficiently used by several authors; see [9, 10, 24, 26, 27, 28, 29]. The classical normal form theory is appropriately refined by Kang and co-authors to include invertible changes of the control state feedbacks; see [24, 26, 27, 28, 29]. Our approach uses hypernormalization (simplification beyond classical normal forms) of the classical normal forms by applying nonlinear time rescaling and also additional nonlinear transformations taken from the symmetry transformation group of the linearized system. This is new in both theory and applications. As far as theory is concerned, this is a contribution to the orbital and parametric normal form classifications of singularities that fits in a long tradition. As a contribution to applications, the extra hypernormalization process enables the use of and can propose the type of effective nonlinear state feedback multiple-input controller. The latter can be achieved by finding out whether or not a parametric singularity is a universal asymptotic unfolding.

The parametric normal form system (4.11) potentially lays the groundwork for applications in real life problems. Engineering problems are mostly involved with parameters such as control parameters, and it is important (when it is feasible) to find explicit direct transformations that transform the asymptotic unfolding parameters to the original parameters of the system. This provides a tool to do the bifurcation analysis of the problem based on the actual controlling parameters. However, this is only feasible when the original system has enough parameters and has them in the right places such that they can actually play the role of the asymptotic unfolding for the system. Therefore, it is important to identify when and which parametric terms of the original system can effectively play the role of unfolding terms and then remove the redundant parameters. This problem is motivated and greatly influenced by James Murdock and is our most important claimed contribution in this paper. The remainder of this section is devoted to introducing an algorithm for performing this task. This approach potentially proposes certain effective controlling parameters within a parametric system for a possible engineering design. We have implemented our suggested approach in Maple to illustrate that it is computable and works successfully.

Remark 6.1. The bifurcation analysis of the universal asymptotic unfolding normal form system provides all possible real world asymptotic (finitely determined) dynamics of an engineering problem. However, the necessity for adding extra unfolding parameters concludes that the original parametric system may not exhibit all such possible dynamics. Hence, in these circumstances, the desired dynamics may not always be produced by the existing modeling parameters, and modeling refinement is required. Therefore, one needs to find other (already ignored) small parameters in the physics of the problem (our approach provides effective suggestions) to incorporate them in the model for a more comprehensive engineering design.

Denote the $(s+1)$-jet of the vector field (4.11) by

(6.1)
$$\tilde{v}(x,\rho,\nu_1,\dots,\nu_N) := \begin{pmatrix} 2\rho^2 + a_r x^{r+1} + \beta_s x^{s+1} + \sum_{1\le i\le r} \nu_i x^{i-1} + \sum_{i=r+1}^{N} \nu_i x^{k_i+1} \\ -\dfrac{a_r(r+1)}{2} x^r\rho + \frac{1}{2}\beta_s x^s \rho - \sum_{1\le i\le r} \frac{i-1}{2}\nu_i x^{i-2}\rho + \frac{1}{2}\sum_{i=r+1}^{N} \nu_i x^{k_i}\rho \end{pmatrix}$$

and the polynomial map $\nu(\mu)$ and matrix J by

$$\nu(\mu) := (\nu_1(\mu),\dots,\nu_N(\mu)), \qquad J := \left.\frac{\partial(\nu_1,\dots,\nu_N)}{\partial(\mu_1,\dots,\mu_p)}\right|_{\mu=\mathbf{0}}. \tag{6.2}$$

Assume that

$$\operatorname{rank}(J) = k \quad \text{for} \quad k \leq \min\{p, N\}.$$

Then there always exists a linear space M such that

$$\mathbb{R}^p = \ker J \oplus M.$$

Similar to the formal basis style [21, page 1006] in finding complement spaces, we may choose the complement space M such that $M := \operatorname{span}\{e_{\sigma(i)} \,|\, i = 1, \ldots, k\}$ for a permutation $\sigma \in S_p$. Here e_j denotes the standard basis of $\mathbb{R}^p$. Let $\hat{e}_i := Je_{\sigma(i)}$ for $i = 1, \ldots, k$. Thus,

$$\operatorname{range}(J) = \operatorname{span}\{\hat{e}_i \,|\, i = 1, \ldots, k\}.$$

Define $\hat{\mu} := (\mu_{\sigma(k+1)}, \ldots, \mu_{\sigma(p)})$ and the polynomial map $\psi_{\hat{\mu}} : M \to \mathbb{R}^k$ by

$$\psi_{\hat{\mu}}(\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(k)}) = (\nu \cdot \hat{e}_1, \nu \cdot \hat{e}_2, \ldots, \nu \cdot \hat{e}_k), \tag{6.3}$$

where ν is the polynomial map given in (6.2). Since the Jacobian of $\psi_{\mathbf{0}}$ evaluated at the origin has full rank, and assuming that $\hat{\mu}$ is sufficiently small, the map $\psi_{\hat{\mu}}$ is locally invertible. Then

$$\big(\mu_{\sigma(1)}(\nu, \hat{\mu}), \mu_{\sigma(2)}(\nu, \hat{\mu}), \ldots, \mu_{\sigma(k)}(\nu, \hat{\mu})\big) = {\psi_{\hat{\mu}}}^{-1}\big(y_1(\nu), \ldots, y_k(\nu)\big), \tag{6.4}$$

where $y_i(\nu) = \nu \cdot \hat{e}_i$ for $i = 1, \ldots, k$. Combining the map given by (6.4) and ν given by (6.2), the following proposition holds.

Proposition 6.2. *Assume that* $\operatorname{rank}(J) = k$*, and consider* $\tilde{v}$ *in* (6.1)*, the permutation* $\sigma \in S_p$ *described above, and invertible reparametrizations* $\mu_{\sigma(i)}(\nu, \hat{\mu})$ *(for* $i = 1, \ldots, k$*) given by* (6.4)*. Then there exist polynomial functions* $\nu_{\sigma(i)}(\mu)$ *(for* $i = k+1, k+2, \ldots, N$*) so that* (4.1) *is equivalent to*

$$[\dot{x}, \dot{\rho}] = \tilde{v}\left(x, \rho, \mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(k)}, \nu_{\sigma(k+1)}(\mu), \ldots, \nu_{\sigma(N)}(\mu)\right), \tag{6.5}$$

that is, the s-universal asymptotic unfolding planar normal form.

Proof. Given Theorem 4.1, and thanks to (6.4), a straightforward computation shows that (4.1) takes the form of (6.5); see also the proof of [22, Lemma 3.3]. ■

We refer to the parameters $\mu_{\sigma(1)}, \mu_{\sigma(2)}, \ldots, \mu_{\sigma(k)}$ as *distinguished parameters*; i.e., they play the role of universal asymptotic unfolding parameters. Adding extra asymptotic unfolding parameters to the system is only justified when the original control parameters of the system cannot fully do the unfolding job. In this case, the control parameters may still have influences upon the added unfolding parameters, and any such relation is useful for their applications in bifurcation control and needs to be computed. This is computed through Proposition 6.2. The map given by (6.4) projects the transition sets associated with the planar differential system (6.1) from the ν-variables into the original variables $\mu_{\sigma(i)}$ for $i = 1, \ldots, k$.

7. Illustrating examples with two imaginary uncontrollable modes. This paper is concerned with designs of nonlinear controllers for linearly uncontrollable systems; i.e., linear uncontrollability means that linear controllers are not sufficient to force an initial state to another state in a finite time, and thus nonlinear controllers are instead designed to control its dynamics. We emphasize that we do not treat control systems, but we indeed provide cognitive suggestions for designing controllers for linearly uncontrollable differential systems. Kang and Krener [28] described an extended Brunovsky canonical form for *linearly controllable nonlinear control systems*; see [28, Theorems 1 and 3] and [28, equation (2.11a)]. This latter equation in [28] accommodates a single zero singularity among its simplest examples, while at least two imaginary uncontrollable modes are needed for a linearly uncontrollable system. This justifies Hopf-zero singularity amongst the simplest examples of a nonlinear system that is not linearly controllable. Hopf bifurcation control is readily available given our results in [21, 22]. The Bogdanov–Takens bifurcation controller design is a project in progress.

We have developed a Maple program to compute the parametric normal forms for any small perturbation of a family of Hopf-zero singularity (Case I). Further, it may take constant coefficients (different from perturbation parameters) of the parametric systems as unknown symbols rather than merely taking numerical coefficients. This greatly promotes its potential for practical applications. Our program will be updated as our research progresses, aiming at integrating and enhancing our results [17, 18, 19, 20, 21, 22] into a user-friendly Maple library for normal form analysis of singularities.

Any control system [24, equations (2.1)–(2.2)] on a three dimensional central manifold with two imaginary uncontrollable modes can be transformed into

$$\dot{x} := u + f(x, z_1, z_2, u), \quad \dot{z_1} := z_2 + g(x, z_1, z_2, u), \quad \dot{z_2} := -z_1 + h(x, z_1, z_2, u), \tag{7.1}$$

using linear changes in state and feedback variables; see [24, equation (2.3)]. Here u stands for a quadratic multiple-feedback controller and we assume that it is given by

$$u := \mu_1 + \mu_2 z_2 + \mu_3 z_1 + \mu_4 x + \mu_5 {z_2}^2 + \mu_6 z_1 z_2 + \mu_7 x z_2 + \mu_8 x z_1, \tag{7.2}$$

where μ_i's stand for the control parameters.

7.1. The case $r := 1$.

We assume that

$$f := -d_1 x^2 + d_2 {z_1}^2 + d_3 x^3, \quad g := d_1 x z_1, \quad h := d_1 x z_2, \quad d_1 d_2 d_3 \neq 0. \tag{7.3}$$

Our approach can be applied to similar examples. A universal asymptotic unfolding normal form (5.6) is described by

$$a_1 = -\frac{4d_1}{d_2}, \qquad \beta_2 = \frac{3d_3}{d_2},$$

and

$$\nu_1(\mu) = \frac{4}{d_2}\mu_1 - \frac{8}{{d_2}^2}\mu_1\mu_5 - \frac{13d_1}{2d_2}{\mu_1}^2 + \frac{1}{2d_1d_2}{\mu_4}^2 + \frac{1}{2}\frac{{d_1}^2}{d_2d_3}\mu_1\mu_4 + \frac{1679}{3456}\frac{{d_3}^2{\mu_1}^2}{d_2{d_1}^3} + \frac{365}{64}\frac{d_3\mu_1\mu_4}{{d_1}^2d_2},$$
$$\nu_2(\mu) = \frac{2}{d_2}\mu_4 - \frac{2}{{d_2}^2}\mu_4\mu_5 + \frac{{d_1}^2}{8d_2d_3}{\mu_4}^2 - \frac{2d_1}{d_2}\mu_1\mu_4 + \frac{1271}{768}\frac{d_3{\mu_4}^2}{{d_1}^2d_2} + \frac{15419}{13824}\frac{{d_3}^2\mu_1\mu_4}{d_2{d_1}^3}.$$

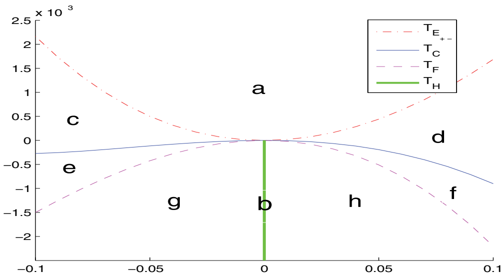


Figure 7. *The 2-jet estimated varieties in terms of original distinguished parameters (μ_1, μ_4) in (7.1). The vertical and horizontal axes are μ_1 and μ_4, respectively.*

Using our Maple program, the distinguished parameters are μ_1 and μ_4. Therefore, we choose $\mu_j = 0$ for any $j \neq 1, 4$. Then the 2-jet approximations for the inverse maps ${\nu_1}^{-1}$ and ${\nu_2}^{-1}$ are governed by

$$\begin{aligned}\mu_1(\nu_1,\nu_2) :=& \frac{1}{4}d_2\nu_1 + \frac{13d_1{d_2}^2}{128}{\nu_1}^2 - \frac{{d_2}^2}{32d_1}{\nu_2}^2 - \frac{365}{2048}\frac{{d_2}^2d_3\nu_1\nu_2}{{d_1}^2} - \frac{1}{64}\frac{{d_2}^2{d_1}^2\nu_1\nu_2}{d_3}\\ &- \frac{1679}{221184}\frac{{d_2}^2{d_3}^2{\nu_1}^2}{{d_1}^3},\\ \mu_4(\nu_1,\nu_2) :=& \frac{d_2}{2}\nu_2 + \frac{d_1{d_2}^2}{8}\nu_1\nu_2 - \frac{1271}{6144}\frac{{d_2}^2d_3{\nu_2}^2}{{d_1}^2} - \frac{{d_1}^2{d_2}^2}{64d_3}{\nu_2}^2 - \frac{15419}{221184}\frac{{d_2}^2{d_3}^2\nu_1\nu_2}{{d_1}^3}.\end{aligned}$$

For numerical simulation, we choose

$$d_1 := -1, \quad d_2 := 4, \text{ and } d_3 := \frac{4}{3}.$$

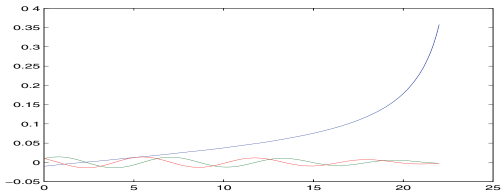
Then $a_1 := 1$ and $\beta_2 := 1$. Thus, a 3-jet estimation in terms of μ_1 and μ_4 for the bifurcation varieties $T_{E_\pm}$ and T_C is given by

$$\begin{aligned}T_{E_\pm} &:= \left\{\left(\mu_1, \frac{3}{16}{\mu_4}^2 - \frac{1061}{4608}{\mu_4}^3 + \frac{3486907}{7077888}{\mu_4}^4 - \frac{31594532951}{36691771392}{\mu_4}^5 + \frac{29226779813459}{21134460321792}{\mu_4}^6\right)\right\},\\ T_C &:= \left\{\left(\mu_1, -\frac{1}{16}{\mu_4}^2 - \frac{157}{512}{\mu_4}^3 + \frac{22826407}{63700992}{\mu_4}^4 - \frac{347712683}{452984832}{\mu_4}^5 + \frac{92683866456661}{63403380965376}{\mu_4}^6\right)\right\},\\ T_F &:= \left\{\left(\mu_1, -\frac{3}{16}{\mu_4}^2 - \frac{1589}{4608}{\mu_4}^3 + \frac{4797371}{21233664}{\mu_4}^4 - \frac{23112314429}{36691771392}{\mu_4}^5 + \frac{8519077413067}{7044820107264}{\mu_4}^6\right)\right\},\end{aligned}$$

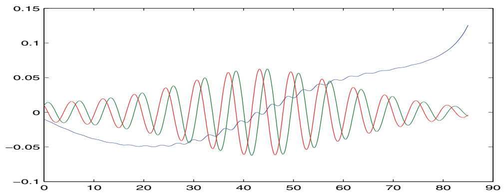
and $T_H := \{(\mu_1, 0)\}$; see Figure 7. To validate our approximated transition sets, we choose the following eight values from regions (a)–(h), respectively:

$$\begin{aligned}(\mu_1,\mu_4) := &(0.004, 0.001), (-0.004, 0.001), (5\times 10^{-4}, -0.1), (5\times 10^{-4}, 0.08),\\ &(-0.001, -0.08), (-0.001, 0.08), (-1.5\times 10^{-3}, -0.06), (-1.5\times 10^{-3}, 0.001).\end{aligned}$$

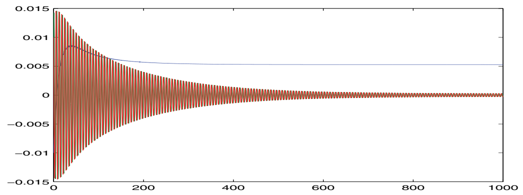
From numerical simulations using MATLAB and the initial point $(x, y, z) = (-0.01, 0.01, 0.01)$, we accordingly obtain Figures 8(a)–(g). To observe an attracting invariant torus, we instead



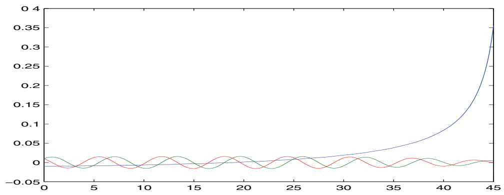
(a) Region a: Asymptotic to the x-axis, the solution approaches infinity for $\mu_1 = 0.004, \mu_4 = 0.001$.



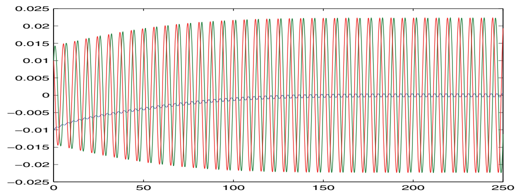
(b) Parameters $\mu_1 = -0.004$ and $\mu_4 = 0.001$ are taken from region b. Note that here $d_3 := \frac{3}{4}$.



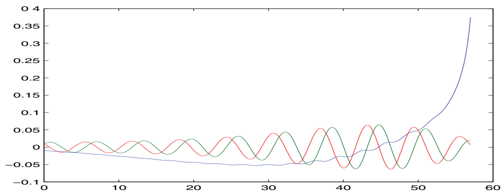
(c) Region c: The orbit converges to a stable equilibrium for $\mu_1 = 0.0005$, $\mu_4 = -0.1$; i.e., E_- is a sink.



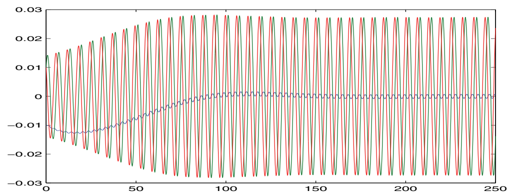
(d) Region d: For $\mu_1 = 0.0005$, $\mu_4 = 0.08$, the equilibrium E_- is a saddle, while E_+ is a source point.



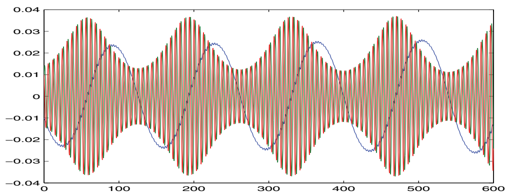
(e) The orbit approaches a stable limit cycle (corresponding to C) in three dimensions for $\mu_1 = -0.001$, $\mu_4 = -0.08$ from region e.



(f) The orbit diverges to infinity while rotating around the x-axis. Here C is a source and $\mu_1 = -0.001$, $\mu_4 = 0.08$ from region f.



(g) Region g: The solution rotates and converges to a stable limit cycle C for $\mu_1 = -0.0015$, $\mu_4 = -0.06$.



(h) Region h: An attracting invariant torus for $d_3 := -\frac{3}{4}$ and $\mu_1 = -0.0015$, $\mu_4 = 0.001$.

Figure 8. *Time series $(x(t), y(t), z(t))$ for different choices of parameters from regions* (a)–(h) *in Figure 7.*

choose $d_3 := -\frac{4}{3}$; see Figure 8(h). These are compatible with our anticipated bifurcations and demonstrate that cognitive choices for different values of (μ_1, μ_4) from connected components of Figure 7 effectively control the local dynamics of the system (7.1).

7.2. The case $r := 2$ and $s := 3$. Consider the control system given by (7.1), (7.3), and

$$f := d_2 {z_1}^2 + d_3 x^3 + d_4 x^4, \quad g := u - \frac{3}{2} d_3 x^2 z_1, \quad h := -\frac{3}{2} d_3 x^2 z_2, \qquad d_2 d_3 d_4 \neq 0, \tag{7.4}$$

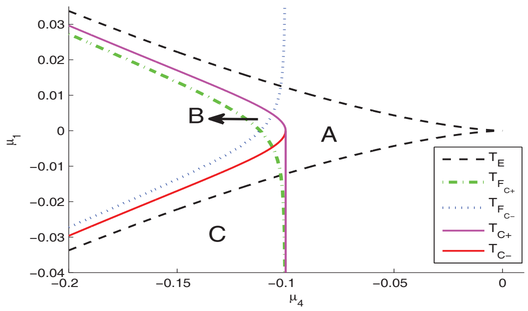


Figure 9. *Transition varieties for the system* (7.1), (7.3), (7.4), $(r, s) := (2, 3)$, *and* $\mu_3 + \mu_4 := -0.1$. *Region* B *shows the region above* T_{FC_+} *and below* T_{C_+}.

whose transition varieties are depicted in Figure 9. Using our Maple program, the distinguished parameters are chosen as μ_1, μ_3, and μ_4, and in the parametric normal form (5.11), $a_2 := \frac{4d_3}{d_2}$, $\beta_3 := \frac{16d_4}{5d_2}$,

$$(\nu_1, \nu_2, \nu_3) := \left(\frac{2}{d_2}(2\mu_1 + \mu_1\mu_3\mu_4), \frac{2}{d_2}\left(\mu_4 - \mu_3 + 2\mu_3{\mu_4}^2\right), \frac{2}{d_2}(\mu_3 + \mu_4)\right),$$

and

$$(\mu_1, \mu_2, \mu_3) := \left(\frac{d_2}{4}\nu_1 - \frac{{d_2}^3}{128}\nu_1{\nu_4}^2 + \frac{{d_2}^3}{128}\nu_1\nu_3^2, \frac{d_2}{4}\nu_4 - \frac{d_2}{4}\nu_3, \frac{d_2}{4}\nu_4 + \frac{d_2}{4}\nu_3\right).$$

Hence, estimated transition sets are given by

$$T_E := \left\{(\mu_1, \mu_3, \mu_4) : \frac{432{\mu_1}^2(1 + \mu_3\mu_4 + 0.25{\mu_3}^2{\mu_4}^2)}{{d_2}^2} + \frac{256{\mu_4}^3(1 + 3\mu_3\mu_4 + 3{\mu_3}^2{\mu_4}^2 + {\mu_3}^3{\mu_4}^3)}{{d_2}^3}\right\},$$

$$T_{C_\pm} := \left\{\left(\frac{\mp 4\sqrt{3\mu_3(1 - {\mu_4}^2)}}{9\sqrt{d_2}}\,\frac{\mu_3 + 3\mu_4 + 2\mu_3{\mu_4}^2}{2 + \mu_3\mu_4}, \mu_3, \mu_4\right) : d_2\mu_3 \geq 0\right\} \cup \{(\mu_1, 0, \mu_4) : d_2\mu_1 \leq 0\},$$

$$T_{FC_\pm} := \left\{(\mu_1, \mu_3, \mu_4) : \sqrt{\mu_3}\mu_1 = \mp\frac{11{\mu_3}^2 + 30\mu_3\mu_4 + 3{\mu_4}^2 - 24\mu_3{\mu_4}^3 + 8{\mu_3}^2{\mu_4}^2 - 16{\mu_3}^2{\mu_4}^4}{6\,\mathrm{sign}(d_2)\sqrt{3d_2(1 - {\mu_4}^2)}(2 + \mu_3\mu_4)}\right\},$$

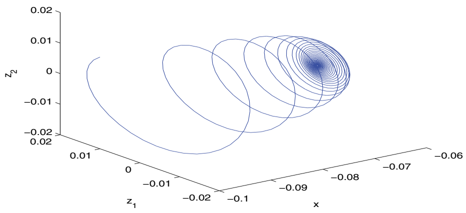
and

$$T_H := \{(\mu_1, \mu_3, -\mu_3)\}.$$

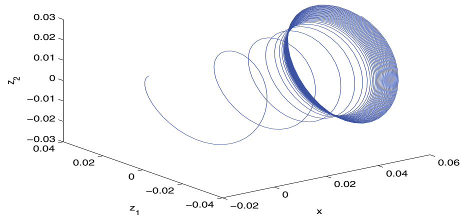
For numerical simulations, we take $d_2 := 4, d_3 := 1, d_4 := \frac{5}{4}$, and $\mu_3 + \mu_4 := -0.1$, and we take the following values for parameters (μ_1, μ_4),

$$(-0.0047, -0.112), (-0.005, -0.08), (-0.035, -0.15), (-0.035, -0.15),$$

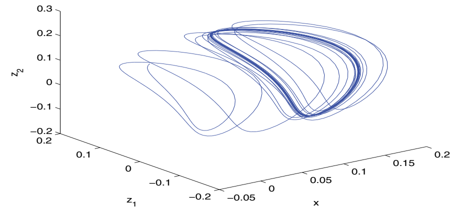
and obtain Figures 10(a)–(d), respectively. Figures 10(c)–(d) suggest the coexistence of two distinct attractors, that is, a bistability involving equilibria and limit cycles in region C of Figure 9.



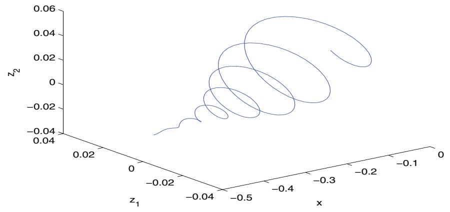
(a) Region A: An orbit approaching a stable equilibrium E_1.



(b) Region B: An orbit approaching a stable limit cycle C_+.



(c) Region C: An orbit approaching and rotating around the limit cycle C_+.



(d) Region C: An orbit approaching a stable equilibrium.

Figure 10. *Orbits associated with parameters of the connected regions* A, B, *and* C *in Figure* 9. *Figures* 10(c)–(d) *demonstrate a bistability: Two orbits associated with the same parameters from region* C *in Figure* 9 *approach two different attractors. Orbits in* 10(c)–(d) *start with* $(x := 0.01, z_1 := 0.1, z_2 := 0.1)$ *and* $(x := -0.1, z_1 := 0.01, z_2 := 0.01)$, *respectively.*

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