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J. Differential Equations 267 (2019) 407–442

Journal of Differential Equations

www.elsevier.com/locate/jde

Vector potential normal form classification for completely integrable solenoidal nilpotent singularities ☆,☆☆

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Received 30 August 2018
Available online 26 January 2019

Abstract

We introduce an $\mathfrak{sl}_2$-invariant family of polynomial vector fields with an irreducible nilpotent singularity. In this paper, we are concerned with characterization and normal form classification of these vector fields. We show that the family is a Lie subalgebra and each vector field from this family is volume-preserving, completely integrable, and rotational. All such vector fields share a common quadratic invariant. We provide a Poisson structure for the Lie subalgebra from which the second invariant for each vector field can be readily derived. We show that each vector field from this family can be uniquely characterized by two alternative representations which can be found in applications: one uses a vector potential while the other uses two functionally independent Clebsch potentials. Our normal form results are designed to preserve these structures and representations.

MSC: 34C20; 34A34

☆ This research is part of the PhD dissertation of the second author (under supervision of the first author) at Isfahan University of Technology.

☆☆ This research was in part supported by a grant from IPM (No. 96370420).

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https://doi.org/10.1016/j.jde.2019.01.016

Keywords: Normal form classification; Triple zero singularity; $\mathfrak{sl}_2$-Lie algebra representation; Clebsch potentials; Completely integrable vector field

1. Introduction

We are concerned with the nonlinear normal form classification of an $\mathfrak{sl}_2$-Lie algebra generated family of vector fields with a nilpotent linear part, *i.e.,* $-\mathrm{N} := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z}$. The Jacobson–Morozov theorem [21, Chapter X, Section 2] provides the other two generators that form an $\mathfrak{sl}_2$-triple along with N. Consider N as a differential operator. Then by the adjoint action of the $\mathfrak{sl}_2$-Lie algebra on (nonlinear) vector fields, we introduce an $\mathfrak{sl}_2$-invariant family of vector fields. We show that the set of all such nonlinear vector fields whose linear part is a multiple scalar of N constitutes a Lie algebra, where we denote it by $\mathscr{B}$. Recall that in the two-dimensional case this family consisted of the Hamiltonian systems [2]. This paper is part of a major project, that is to obtain the normal form classification of the general three dimensional irreducible nilpotent singularities.

An important goal in our normal form results is to detect, compute and preserve possible symmetries and geometrical features of a vector field's flow. Hence, several geometric properties for the $\mathscr{B}$-family are carefully studied in this paper. The analysis of such vector fields must respect these geometrical features which have applications in different applied disciplines. However, the classical normal form computations typically destroy certain symmetries of the truncated normal form system. These may include the system's properties such as volume-preserving, Clebsch potentials, vector potentials, etc. Thus in either of these cases, the dynamics analysis of the truncated normal form is not appropriate. Hence it is important to use permissible transformations which preserve the system's symmetry; also see [16,18]. An automatic consequence here of the Lie algebraic approach is that our (truncated) normal form results preserve all the different representations (described below) in the paper such as vector potential, Clebsch potentials, and the volume-preserving property.

Solenoidal (volume-preserving) vector fields appear in disciplines such as magnetic fields and fluid mechanics; *e.g.,* see [5,20,23–25]. Computing the invariants of vector fields in the $\mathscr{B}$-family is an important goal in this paper. Any solenoidal vector field, say v, takes a vector potential representation, that is, there exists a vector field, say w, whose curl is v, *i.e.,* $\nabla \times w = v$. We prove that all vector fields in $\mathscr{B}$ are solenoidal and provide the method and formulas for deriving their vector potential normal forms. We further introduce a Poisson algebra that is Lie-isomorphic to $\mathscr{B}$ through a Lie isomorphism ψ. In immediate important consequence is that the Lie isomorphism ψ associates a first integral $\psi(v)$ to each vector field v from $\mathscr{B}$. We further show that the quadratic polynomial $\Delta := xz - y^2$ is a second first integral for all vector fields in $\mathscr{B}$.

Analytic normalization of an analytic vector field has close relations with complete integrability of the vector field; *e.g.,* see [37,38,45,49]. We recall that two first integrals for v in $\mathscr{B}$ are called *functionally independent* when their gradients have a rank of 2 for almost everywhere; *e.g.,* see [45, page 3553] and [37]. The level curves of these invariants provide a comprehensive understanding about the orbits in the state space associated with the vector field's flow. Each vector field v from the $\mathfrak{sl}_2$-invariant Lie algebra $\mathscr{B}$ is a completely integrable solenoidal vector field; *i.e.,* we show that the invariants Δ and $\psi(v)$ for each $v \in \mathscr{B}$ are functionally independent. There is another alternative representation for completely integrable solenoidal vector fields, that is given by the two functionally independent invariants. Indeed, we prove that each vector field v

in $\mathscr{B}$ equals the exterior product of the gradients of Δ and $\psi(v)$; the latter is obtained through the Lie isomorphism ψ between $\mathscr{B}$ and our introduced Poisson algebra. The first integrals in the exterior product are referred as Clebsch potentials or Euler potentials of the vector field v; *e.g.,* see [23,24,27]. We refer to Δ by the primary Clebsch potential and $\psi(v)$ as the secondary Clebsch potential for v. We further conclude that these families of triple zero singularities are rotational vector field, that is, their curl is non-zero. This implies that these are not gradient vector fields.

Finally, we prove that $\mathscr{B}$ is the set of all multiple scalars of solenoidal vector fields such as v, that is given by

$$v := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \mathrm{v}(x, y, z), \tag{1.1}$$

where $\mathrm{v} : \mathbb{R}^3 \to \mathbb{R}^3$ denotes a vector field without constant and linear part,

$$\mathrm{div}(v) = 0, \qquad \text{and} \qquad v(\Delta) = 0. \tag{1.2}$$

The algebraic results throughout this paper work fine for any field with zero characteristic. However, we merely present them as for the field $\mathbb{R}$, since this we work with real differential equations. Note that at our convenience we mix freely the use of notations and terminologies such as vector fields, differential systems, and differential operators. Note that x and Δ are the two generators of the invariant algebra for the linear vector field N. We refer to the vector field v in equations (1.1)–(1.2) as a completely integrable system since it has two functionally independent invariants Δ and $\psi(v)$. The Poisson structure and the Lie isomorphism ψ provide a practical method for deriving the second first integral within the invariant algebra of vector fields given by (1.1)–(1.2) and their normal forms.

Normal form classification of nilpotent singularities has been (and still is!) a challenging task. Even in the two-dimensional case, there have been numerous important contributions in various types and approaches; *e.g.,* see [1–3,6,7,12,15,19,22,39–42,44,46,48]. There have only been a few contributions in three dimensional state space cases; see [13,47] where hypernormalization is performed up to degree three; also see [17] and [30–34]. In this paper we provide a complete normal form classification for all vector fields v in equations (1.1)–(1.2), that is, the set of all completely integrable solenoidal nilpotent singularities where Δ is one of their invariants and a multiple scalar of N is their linear part. These vector fields and their normal forms are uniquely characterized by their secondary Clebsch potential. Indeed, the primary Clebsch potential Δ is always preserved throughout the normalization steps while the normalizing transformations naturally reflect the normal form changes into the secondary Clebsch potential. In Theorem 5.1, we prove that a vector field given by (1.1)–(1.2) can be either linearized or uniquely transformed into the formal normal form vector field

$$\begin{aligned}\mathbf{w} := &-x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + z^p(z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}) \\ &+ \sum_{k=p+1}^{\infty}\sum_{i=0}^{[\frac{k+p}{2}]} b_{i,k} z^i (xz - y^2)^{k-2i+p}(z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}),\end{aligned} \tag{1.3}$$

where $b_{i,k} \in \mathbb{R}$ and p is a natural number. In addition, the secondary Clebsch potential normal form is given by

$$\mathbf{I}(x,y,z) = x + \frac{1}{p+1}z^{p+1} + \sum_{k=p+1}^{\infty}\sum_{i=0}^{[\frac{k+p}{2}]}\frac{b_{i,k}}{i+1}z^{i+1}(xz-y^2)^{k-2i+p}. \tag{1.4}$$

The normal form invariant (1.4) can sometimes be used for further reduction of the normal form vector fields; *e.g.,* see Section 5.

Now we describe the organization of the rest of this paper. We introduce a family of $\mathfrak{sl}_2$-invariant irreducible vector spaces of vector fields in Section 2. We further prove that this family constitutes a Lie algebra and derive the associated structure constants. Section 3 is devoted to the introduction of a Poisson algebra and the proof that it is Lie isomorphic to $\mathscr{B}$. Next, we discuss the geometrical properties of the $\mathscr{B}$-family in Section 4. In particular, we show that our $\mathfrak{sl}_2$-invariant introduced family of vector fields are fully characterized by equations (1.1)–(1.2). Two further representations for each such vector field are presented in this section by using their Clebsch potentials and vector potentials. Section 5 is dedicated to study the normal form classification for vector fields (1.1)–(1.2). Some practical formulas for normal form coefficients of up to degree three for a given triple zero singularity (1.1)–(1.2) are presented.

2. Algebraic structures

Let

$$\mathrm{N} := x\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial z}, \quad \mathrm{M} := z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}, \quad \text{and}$$
$$\mathrm{H} := [\mathrm{M},\mathrm{N}] = \mathrm{MN} - \mathrm{NM} = 2z\frac{\partial}{\partial z} - 2x\frac{\partial}{\partial x}. \tag{2.1}$$

The triple $\{\mathrm{M},\mathrm{N},\mathrm{H}\}$ generates an $\mathfrak{sl}_2$ Lie algebra, *i.e.,*

$$[\mathrm{M},\mathrm{N}] = \mathrm{H}, \qquad [\mathrm{H},\mathrm{M}] = 2\mathrm{M}, \qquad [\mathrm{H},\mathrm{N}] = -2\mathrm{N}.$$

We denote $(\mathrm{N}^n f)\,v$ for the iterative action of N as a differential operator on the scalar function f that is also multiplied with v. Further for a vector field v, $\mathrm{N}^n(v)$ is inductively defined by

$$\mathrm{N}(v) := \mathrm{ad}_{\mathrm{N}}v, \quad \text{and} \quad \mathrm{N}^n(v) := \mathrm{ad}_{\mathrm{N}}\mathrm{N}^{n-1}(v) \qquad \text{for } n > 1.$$

Note that N as an operator distinguishes vector fields from scalar functions: the operator N merely acts on scalar functions as a differential operator while it acts as a Lie operator on vector fields. By [8, Proposition 2], $\mathbb{R}[[z,\Delta]]$ is the invariant ring for M. If for a homogeneous scalar polynomial function $f:\mathbb{R}^3[x,y,z]\to\mathbb{R}$ in $\ker\mathrm{M} = \langle\Delta, z\rangle$, there exists an integer $\omega_f \in \mathbb{Z}$ so that

$$\mathrm{H}f = \omega_f f,$$

then ω_f and f are called the eigenvalue and eigenfunction of the differential operator H, respectively. The algebra of first integrals for M is the same as $\ker\mathrm{M} = \langle\Delta, z\rangle$; see [9], [31, Chapter 2] and [36, Chapter 9] for more details on the representation of $\mathfrak{sl}_2$ and normal form theory. In this section, we use the $\mathfrak{sl}_2$-triple (2.1) to generate a family of irreducible $\mathfrak{sl}_2$-invariant vector spaces.

The set of all such invariant vector spaces constitutes a Lie algebra. This consists of all completely integrable and solenoidal vector fields of triple zero singularities, that is, their linear part is a scalar multiple of N in (2.1). Vector fields from this family can be considered as analogues in three dimensional state space for completely integrable Hamiltonian systems which always require even dimensions in state space; also see [16, page 2812].

Notation 2.1.

- The following notations frequently appear in this paper.

$$\begin{aligned}\sigma_1 &:= \sigma_1(s_1, s_2, k_1, k_2) = s_1 + s_2 + k_1 + k_2,\\ \sigma_2 &:= \sigma_2(q_1, q_2, i_1, i_2) = q_1 + q_2 - i_1 - i_2.\end{aligned} \tag{2.2}$$

- We denote $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ for the standard basis of $\mathbb{R}^3$ and $\kappa_{l,i} := \frac{i!}{(i-l)!}$.
- We use the Pochhammer k-symbol notation for any $a, b \in \mathbb{R}, k \in \mathbb{N}$ as $(a)_b^k := \prod_{j=0}^{k-1}(a + jb)$.
- Throughout this paper we frequently use some constants or variables with negative powers in the denominator (or numerator) of a fraction. The reader should merely treat this as a formal misuse of notation to shorten the formulas.

Now we present some technical results which play a central role in this paper.

Lemma 2.2. *Let f be a homogeneous scalar polynomial function. Then,*

$$\mathrm{N}^n(f\mathrm{M}) = \mathrm{N}^n(f)\mathrm{M} - n\mathrm{N}^{n-1}(f)\mathrm{H} - n(n-1)\mathrm{N}^{n-2}(f)\mathrm{N}, \quad \textit{for any } n \in \mathbb{N}_0. \tag{2.3}$$

Proof. The proof is by induction. For $n = 1$, we have

$$\mathrm{N}(f\mathrm{M}) = \mathrm{N}(f)\mathrm{M} + f[\mathrm{N}, \mathrm{M}] = \mathrm{N}(f)\mathrm{M} - f\mathrm{H} = \mathrm{N}(f)\mathrm{M} - f\mathrm{H}.$$

By the induction hypothesis we have

$$\begin{aligned}&\mathrm{N}^{n+1}(f\mathrm{M})\\ &\quad= [\mathrm{N}, \mathrm{N}^n(f)\mathrm{M} - n\mathrm{N}^{n-1}(f)\mathrm{H} - n(n-1)\mathrm{N}^{n-2}(f)\mathrm{N}]\\ &\quad= \mathrm{N}^{n+1}(f)\mathrm{M} + \mathrm{N}^n(f)\mathrm{NM} - n\mathrm{N}^n(f)\mathrm{H} - n\mathrm{N}^{n-1}(f)\mathrm{NH} - n(n-1)\mathrm{N}^{n-1}(f)\mathrm{N}\\ &\qquad - n(n-1)\mathrm{N}^{n-2}(f)\mathrm{NN} - (\mathrm{N}^n(f)\mathrm{MN} - n\mathrm{N}^{n-1}(f)\mathrm{HN} - n(n-1)\mathrm{N}^{n-2}(f)\mathrm{NN})\\ &\quad= \mathrm{N}^{n+1}(f)\mathrm{M} + \mathrm{N}^n(f)[\mathrm{N}, \mathrm{M}] - n\mathrm{N}^n(f)\mathrm{H} - n\mathrm{N}^{n-1}(f)[\mathrm{N}, \mathrm{H}] - n(n-1)\mathrm{N}^{n-1}(f)\mathrm{N}\\ &\quad= \mathrm{N}^{n+1}(f)\mathrm{M} - (n+1)\mathrm{N}^n(f)\mathrm{H} - n(n+1)\mathrm{N}^{n-1}(f)\mathrm{N}.\end{aligned}$$

This proves the statement; also see [9, Proposition 1]. □

There is an important corollary to this lemma.

Corollary 2.3. *For any H-eigenfunction $f \in \ker \mathrm{M}$, with eigenvalue ω_f we have*

$$\mathrm{N}^n(f\mathrm{M}) = \frac{2(\omega_f - n + 2)\kappa_{n,\omega_f+2}\mathrm{N}^{n+1}z(f)}{(\omega_f + 2)\kappa_{n+1,\omega_f+2}}\frac{\partial}{\partial x} + \frac{(\omega_f - 2n + 2)\kappa_{n,\omega_f+2}\mathrm{N}^{n}z(f)}{(\omega_f + 2)\kappa_{n,\omega_f+2}}\frac{\partial}{\partial y} - \frac{2n\kappa_{n,\omega_f+2}\mathrm{N}^{n-1}z(f)}{(\omega_f + 2)\kappa_{n-1,\omega_f+2}}\frac{\partial}{\partial z}. \tag{2.4}$$

Lemma 2.4. *For each $l \in \mathbb{N}_0$, the following equalities hold:*

$$\frac{\partial}{\partial x}\mathrm{N}^l = l(l-1)\mathrm{N}^{l-2}\frac{\partial}{\partial z} + l\mathrm{N}^{l-1}\frac{\partial}{\partial y} + \mathrm{N}^l\frac{\partial}{\partial x},$$
$$\frac{\partial}{\partial y}\mathrm{N}^l = 2l\mathrm{N}^{l-1}\frac{\partial}{\partial z} + \mathrm{N}^l\frac{\partial}{\partial y},$$
$$\frac{\partial}{\partial z}\mathrm{N}^l = \mathrm{N}^l\frac{\partial}{\partial z}.$$

Proof. The proof is by induction on l. For instance, by the induction hypothesis we have

$$\frac{\partial}{\partial y}\mathrm{N}^{l+1} = 2l\mathrm{N}^{l-1}\frac{\partial}{\partial z}\mathrm{N} + \mathrm{N}^l\frac{\partial}{\partial y}\mathrm{N} = 2l\mathrm{N}^l\frac{\partial}{\partial z} + 2\mathrm{N}^l\frac{\partial}{\partial z} + \mathrm{N}^l\mathrm{N}\frac{\partial}{\partial y} = 2(l+1)\mathrm{N}^l\frac{\partial}{\partial z} + \mathrm{N}^{l+1}\frac{\partial}{\partial y}.$$

This proves the second equality. □

Lemma 2.5. *Let $q = 2s + r$, where $r = 0$ or $r = 1$ and $s \in \mathbb{N}_0$. Then,*

$$\mathrm{N}^q(z^i) = \sum_{n=0}^{s} \eta_n^{q,i} x^{s-n} y^r z^{i-s-r-n}\Delta^n,$$
$$\mathrm{N}^q(z^i) = \sum_{n=0}^{s} \zeta_n^{q,i} x^{s-n} y^{2n+r} z^{i-n-s-r}, \tag{2.5}$$

where

$$\eta_n^{q,i} := \frac{(-1)^n (s)_{-1}^n (i)_{-1}^{s+n+r}(2i-1)_{-2}^s 2^{s+n+r}}{n!(2i-1)_{-2}^n}, \tag{2.6}$$
$$\zeta_n^{q,i} := \frac{i!(2s+r)!2^{2n+r}}{(s-n)!(2n+r)!(i-n-s-r)!},$$

for $i \geq s + n + r$, while $\eta_n^{q,i} := \zeta_n^{q,i} := 0$ for $i < q + n - s$.

Proof. The proof is straightforward by an induction on q. □

Lemma 2.5 enables us to write $\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j)$ in terms of x, y, z, and Δ.

Proposition 2.6. *Let $q_1 = 2s_1 + r_1$ and $q_2 = 2s_2 + r_2$, where $r_1, r_2 \in \{0, 1\}$ and $s_1, s_2 \in \mathbb{N}_0$. Then,*

$$\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j) = \sum_{n=0}^{\min\{s_1+s_2-\sigma_2, s_1+s_2\}} \tilde{\eta}^{q_1,q_2}_{n,i,j} x^{s_1+s_2-n} y^{r_1+r_2} z^{i+j-(s_1+s_2+r_1+r_2+n)} \Delta^n, \quad (2.7)$$

and

$$\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j) = \sum_{n=0}^{\min\{s_1+s_2-\sigma_2, s_1+s_2\}} \tilde{\zeta}^{q_1,q_2}_{n,i,j} x^{s_1+s_2-n} y^{2n+r_1+r_2} z^{i+j-(s_1+s_2+r_1+r_2+n)},$$

hold where

$$\tilde{\zeta}^{q_1,q_2}_{n,i,j} := \sum_{r=0}^{n} \zeta^{q_1,i}_r \zeta^{q_2,j}_{n-r},$$

$$\tilde{\eta}^{q_1,q_2}_{n,i,j} := \sum_{r=0}^{n} \eta^{q_1,i}_r \eta^{q_2,j}_{n-r}.$$

Further, $\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j) = 0$ when either $s_1 < q_1 - i$ or $s_2 < q_2 - j$.

Proof. This follows from Lemma 2.5. □

The following theorem provides an alternative formula for the expansion of $\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j)$.

Theorem 2.7. *Let $q_1 = 2s_1 + r_1$, $q_2 = 2s_2 + r_2$. Then,*

$$\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j) = \sum_{p=\max\{\sigma_2,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} C^{q_1,q_2}_{p,i,j} \mathrm{N}^{2p+|r_2-r_1|}(z^{2p-\sigma_2+|r_2-r_1|}) \Delta^{s_1+s_2-p+\lfloor\frac{r_1+r_2}{2}\rfloor}, \quad (2.8)$$

where

$$C^{q_1,q_2}_{p,i,j} = \sum_{r=0}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor-p} \frac{\left(\tilde{\eta}^{q_1,q_2}_{r,i,j} - \lfloor\frac{r_1+r_2}{2}\rfloor \tilde{\eta}^{q_1,q_2}_{r-1,i,j}\right)(p+1)_1^{s_1+s_2-p-r+\lfloor\frac{r_1+r_2}{2}\rfloor}}{\eta_0^{2p+|r_2-r_1|,2p-\sigma_2}(s_1+s_2-p-r)!2^{p+r-(s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor)}}$$
$$\times \frac{(p-\sigma_2+1-|r_2+r_1|)_1^{s_1+s_2-p-r+\lfloor\frac{r_1+r_2}{2}\rfloor}}{(4p-2\sigma_2+3-4\lfloor\frac{r_1+r_2}{2}\rfloor)_2^{s_1+s_2-p-r+\lfloor\frac{r_1+r_2}{2}\rfloor}}. \quad (2.9)$$

Proof. See Appendix A. □

Lemma 2.8. *The ring of polynomials invariants of* M *is the polynomial ring* $\mathbb{R}[z, \Delta]$.

Proof. The proof is done in [36], but with a nonstandard choice of nilpotent. Obviously, $\mathbb{R}[z, \Delta]$ is an invariant ring for M. We prove that this is actually the ring of invariants for M using generating functions; see [8–11,31,36]. Following the algorithm and notations used on [8] where d stands for the "degree" and w for the "weight", the generating function for the invariant ring $\mathbb{R}[z, \Delta]$ is given by

$$T(d, w) = \frac{1}{(1 - w^2 d)(1 - d^2)}. \tag{2.10}$$

The invariant z has an eigenvalue (weight) 2 with respect to H and degree 1. For the invariant Δ, the eigenvalue is 0 and its degree is 2. Hence, the term $(1 - w^2 d)$ in the denominator of $T(d, w)$ appears for the invariant z while $(1 - d^2)$ shows up for Δ. Differentiating $wT(d, w)$ with respect to w at $w = 1$ leads to

$$\begin{aligned}\frac{\partial}{\partial w} wT|_{w=1} &= \frac{(1 - w^2 d)(1 - d^2) + 2w^2 d(1 - d^2)}{(1 - w^2 d)^2 (1 - d^2)^2}|_{w=1} \\ &= \frac{(1 - d)(1 - d^2) + 2d(1 - d^2)}{(1 - d)^2 (1 - d^2)^2} = \frac{1}{(1 - d)^3}.\end{aligned}$$

The latter is the generating function for all formal power series associated with three variables. This shows that $\mathbb{R}[z, \Delta]$ equals the kernel of M. □

Theorem 2.9. *Let*

$$V = \operatorname{span}\{\mathrm{N}^n(z^i \Delta^k \frac{\partial}{\partial x}), \mathrm{N}^n(z^i \Delta^k \mathrm{M}), \mathrm{N}^n(z^i \Delta^k \mathrm{E}) \,|n, i, k \in \mathbb{N}_0\},$$

and

$$\mathcal{K} = \operatorname{span}\{z^i \Delta^k \frac{\partial}{\partial x}, z^i \Delta^k \mathrm{M}, z^i \Delta^k \mathrm{E} \,|i, k \in \mathbb{N}_0\},$$

where $\mathbb{N}_0$ denotes nonnegative integers and

$$\mathrm{E} := x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

Then, $\mathcal{K} = \ker \mathrm{ad}_{\mathrm{M}}$ and V is the set of all three dimensional formal vector fields.

Proof. In [36] it is shown how to construct the description (or Stanley decomposition) of the vector fields in $\ker \mathrm{ad}\mathrm{M}$ from the invariants of M (this is what Jim Murdock called *boosting*, [29]). It turns out that every vector field in $\ker \mathrm{ad}\mathrm{M}$ can be written as

$$F_1(z, \Delta]\frac{\partial}{\partial x} + F_2(z, \Delta]\mathrm{M} + F_3(z, \Delta]\mathrm{E}.$$

The generating function is $\frac{(w^2 + w^2 d + d)}{(1 - w^2 d)(1 - d^2)}$ and we leave it to the reader to check that it obeys the Cushman–Sanders test (cf. [8]), that is

$$\left(\frac{\partial}{\partial w} w \frac{(w^2 + w^2 d + d)}{(1 - w^2 d)(1 - d^2)}\right)|_{w=1} = \frac{3}{(1-d)^3}.$$

This proves the Theorem. □

We define

$$\mathsf{B}^l_{i,k} := \frac{1}{\kappa_{l+1,2i+2}} \mathrm{N}^{l+1}(z^i \Delta^k \mathrm{M}), \qquad \text{for} \qquad -1 \le l \le 2i+1, i, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad (2.11)$$

By taking $f := z^i \Delta^k$ in Equation (2.4), $\omega_f = 2i$ and

$$\mathsf{B}^l_{i,k} = \frac{(2i - l + 1)\mathrm{N}^{l+2}(z^{i+1})\Delta^k}{(i+1)\kappa_{l+2,2i+2}} \frac{\partial}{\partial x} + \frac{(i-l)\mathrm{N}^{l+1}(z^{i+1})\Delta^k}{(i+1)\kappa_{l+1,2i+2}} \frac{\partial}{\partial y} - \frac{(l+1)\mathrm{N}^l(z^{i+1})\Delta^k}{(i+1)\kappa_{l,2i+2}} \frac{\partial}{\partial z}. \quad (2.12)$$

Hence, $\mathsf{B}^1_{0,0} := -\mathrm{N}$ and $\mathsf{B}^{-1}_{0,0} := -\mathrm{M}$. Now we introduce $\mathscr{B}$ as the vector space spanned by all nonlinear vector fields from this family with the nilpotent linear part $\mathsf{B}^1_{0,0}$, *i.e.*,

$$\mathscr{B} := \operatorname{span}\left\{\mathsf{B}^1_{0,0} + \sum b^l_{i,k} \mathsf{B}^l_{i,k} \,\middle|\, -1 \leqslant l \leqslant 2i+1, i \in \mathbb{N}, k \in \mathbb{N}_0, b^l_{i,k} \in \mathbb{R}\right\}. \quad (2.13)$$

Note that two more families of vector fields associated with Theorem 2.9 are defined in equations (4.20) and (4.21).

Theorem 2.10. *Denote $\sigma_1(s_1, s_2, k_1, k_2)$ and $\sigma_2(q_1, q_2, i_1, i_2)$ for the indices defined in Equations (2.2). The vector space $\mathscr{B}$ is a Lie algebra with structure constants given by*

$$[\mathsf{B}^{q_1}_{i_1,k_1}, \mathsf{B}^{q_2}_{i_2,k_2}] = \sum_{j=\max\{\sigma_2-1,-1\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} b^{q_1,q_2}_{j,i_1,i_2} \mathsf{B}^{2j+|r_2-r_1|}_{2j-\sigma_2+|r_2-r_1|,\sigma_1-j+\lfloor\frac{r_1+r_2}{2}\rfloor}, \quad (2.14)$$

where

$$b^{q_1,q_2}_{j,i_1,i_2} = \frac{(2j+1-\sigma_2+|r_2-r_1|)\kappa_{2j+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|}}{(2j+|r_2-r_1|+1)} \times \sum_{p=1}^{3} \left(l^{q_2,q_1}_{p,3,i_2,i_1} C^{q_2+3-p,q_1-3+p}_{j,i_2+1,i_1} - l^{q_1,q_2}_{p,3,i_1,i_2} C^{q_1+3-p,q_2-3+p}_{j,i_1+1,i_2}\right),$$

for $2j+1+|r_2-r_1| \neq 0$, while for $j=-1$, $q_1 = 2s_1+1$, and $q_2 = 2s_2$,

$$b^{q_1,q_2}_{-1,i_1,i_2} = \kappa_{0,-2\sigma_2}|r_1-r_2| \sum_{p=1}^{3} \left(l^{q_2,q_1}_{p,2,i_2,i_1} C^{q_2+3-p,q_1-2+p}_{0,i_2+1,i_1} - l^{q_1,q_2}_{p,2,i_1,i_2} C^{q_1+3-p,q_2-2+p}_{0,i_1+1,i_2}\right). \quad (2.15)$$

Here the constants $C^{q_1,q_2}_{p,i,j}$, $l^{q_1,q_2}_{p,3,i_1,i_2}$, $l^{q_1,q_2}_{p,2,i_1,i_2}$, $l^{q_2,q_1}_{p,2,i_2,i_1}$, *and* $l^{q_2,q_1}_{p,3,i_2,i_1}$ *follow Equation* (2.9) *and Equations* (B.1) *in Appendix B.*

Proof. See Appendix B. □

Now we illustrate the structure constants for a few examples.

Example 2.11. The following examples are computed by using a Maple program:

$$
\begin{aligned}
[\mathsf{B}^6_{8,3}, \mathsf{B}^2_{5,2}] &= \frac{1152}{785213}\mathsf{B}^0_{5,9} + \frac{2560}{503217}\mathsf{B}^2_{7,8} - \frac{4256}{38709}\mathsf{B}^4_{9,7} + \frac{1384}{1683}\mathsf{B}^6_{11,6} - \frac{35}{9}\mathsf{B}^8_{13,5},\\
[\mathsf{B}^7_{6,1}, \mathsf{B}^3_{4,1}] &= -\frac{512}{429429}\mathsf{B}^2_{2,6} + \frac{512}{31603}\mathsf{B}^4_{4,5} - \frac{43200}{323323}\mathsf{B}^6_{6,4} + \frac{528}{637}\mathsf{B}^8_{8,3} - \frac{132}{35}\mathsf{B}^{10}_{10,2},\\
[\mathsf{B}^7_{5,6}, \mathsf{B}^6_{7,8}] &= -\frac{224}{347633}\mathsf{B}^3_{2,19} + \frac{27440}{6605027}\mathsf{B}^5_{4,18} - \frac{1400}{138567}\mathsf{B}^7_{6,17} - \frac{18}{299}\mathsf{B}^9_{8,16}\\
&\quad + \frac{91}{100}\mathsf{B}^{11}_{10,15} - \frac{143}{24}\mathsf{B}^{13}_{12,14}.
\end{aligned}
$$

Now we remark that

$$\mathsf{B}^l_{i,k} = \Delta^k \mathsf{B}^l_{i,0},$$

for all nonnegative integers l, i, k; this is due to the equality $\mathrm{N}(\Delta) = 0$. Further recall that Δ is invariant under the $\mathfrak{sl}_2$-action. Yet the following equality demonstrates the complexity of the structure constants:

$$[\mathsf{B}^3_{5,0}, \mathsf{B}^4_{4,0}] = \frac{256}{297297}\mathsf{B}^{-1}_{1,4} - \frac{512}{42471}\mathsf{B}^1_{3,3} + \frac{416}{3927}\mathsf{B}^3_{5,2} - \frac{1312}{1881}\mathsf{B}^5_{7,1} + \frac{10}{3}\mathsf{B}^7_{9,0}.$$

Similar to [3, equations (3.8a)–(3.8h)], we further present some Lie brackets that they are particularly useful for our normal form results:

$$[\mathsf{B}^0_{0,0}, \mathsf{B}^l_{i,k}] = (l-i)\mathsf{B}^l_{i,k}, \tag{2.16}$$

$$[\mathsf{B}^1_{0,0}, \mathsf{B}^l_{i,k}] = (l-2i-1)\mathsf{B}^{l+1}_{i,k}, \tag{2.17}$$

$$[\mathsf{B}^{-1}_{0,0}, \mathsf{B}^l_{i,k}] = (l+1)\mathsf{B}^{l-1}_{i,k}, \tag{2.18}$$

$$[\mathsf{B}^{-1}_{p,0}, \mathsf{B}^q_{i,k}] = \sum_{j=\max\{q-2-i-p,-1\}}^{s-1+\lfloor\frac{r+1}{2}\rfloor} b^{-1,q}_{j,p,i}\mathsf{B}^{2j+|r-1|}_{2j-q+1+i+p+|r-1|,s-1+k-j+\lfloor\frac{r+1}{2}\rfloor}. \tag{2.19}$$

3. Poisson algebra structure

We consider the ring of formal power series $\mathbb{R}[[x, y, z]]$ and define a Poisson bracket on the ring's variables by

$$\{x, y\} = x, \quad \{x, z\} = 2y, \quad \{y, z\} = z. \tag{3.1}$$

Since f and g from $\mathbb{R}[[x, y, z]]$ have each a unique representation as formal power series in x, y, and z, the Leibniz rule and bilinearity of the Poisson bracket are sufficient to uniquely determine Poisson structure for all elements in $\mathbb{R}[[x, y, z]]$. In particular, the Poisson bracket is independent of alternative function multiplications in $\mathbb{R}[[x, y, z]]$, i.e., the Lie bracket of a function f with $g(hk)$ equals to the Lie bracket of f with $(gh)k$. Indeed, by the Leibniz rule we have

$$\begin{aligned}\{f, g(hk)\} - \{f, (gh)k\} = \{f, g\}hk + gk\{f, h\} + gh\{f, k\} - gh\{f, k\}\\ - kg\{f, h\} - kh\{f, g\} = 0.\end{aligned}$$

Hence, the structure constants associated with monomials are given by

$$\begin{aligned}&\left\{x^i y^j z^k, x^m y^n z^p\right\}\\ &\quad = (in + jp - kn - jm)\, x^{i+m} y^{n+j-1} z^{k+p} + 2\,(ip - km)\, x^{i+m-1} y^{n+j+1} z^{k+p-1},\end{aligned} \tag{3.2}$$

for arbitrary nonnegative integers m, n, p, i, j, k. Now define

$$\mathfrak{b}^l_{i,k} := -\frac{\mathrm{ad}_x^{l+1}(z^{i+1}\Delta^k)}{(i+1)\kappa_{l+1,2i+2}}, \qquad \text{for } -1 \le l \le i+1, \text{ and } i, k \in \mathbb{N}_0, \tag{3.3}$$

where $\mathrm{ad}_x f := \{x, f\}$ and $\mathrm{ad}_x^n f := \{x, \mathrm{ad}_x^{n-1} f\}$ for $f \in \mathbb{R}[[x, y, z]]$ and $n > 1$.

Corollary 3.1. *The following formulas provide two alternative polynomial expansions for each $\mathfrak{b}^l_{i,k}$ in terms of x, y, z and Δ:*

$$\mathfrak{b}^{2s}_{i,k} = -\sum_{j=0}^{s} \frac{\eta_j^{2s+1,i+1}}{(i+1)\kappa_{2s+1,2i+2}} x^{s-j} y z^{i-s-j} \Delta^{k+j},$$

$$\mathfrak{b}^{2s}_{i,k} = -\sum_{j=0}^{s} \frac{\zeta_j^{2s+1,i+1}}{(i+1)\kappa_{2s+1,2i+2}} x^{s-j} y^{2j+1} z^{i-j-s} \Delta^{k},$$

$$\mathfrak{b}^{2s+1}_{i,k} = -\sum_{j=0}^{s+1} \frac{\eta_j^{2s+2,i+1}}{(i+1)\kappa_{2s+2,2i+2}} x^{s-j+1} z^{i-s-j} \Delta^{k+j},$$

$$\mathfrak{b}^{2s+1}_{i,k} = -\sum_{j=0}^{s+1} \frac{\zeta_j^{2s+2,i+1}}{(i+1)\kappa_{2s+2,2i+2}} x^{s-j+1} y^{2j} z^{i-j-s} \Delta^{k},$$

where

$$\eta_j^{2s+1,i+1} := \frac{(-1)^j 2^{s+j+1} (s)^j_{-1} (i+1)^{s+j+1}_{-1} (2i+1)^s_{-2}}{(j)!(2i+2)^j_{-2}},$$

$$\zeta_j^{2s+1,i+1} := \frac{2^{2j+1}(i+1)!(2s+1)!}{(s-j)!(2j+1)!(i-j-s)!},$$

$$\eta_j^{2s+2,i+1} := \frac{(-1)^j 2^{s+j+1}(s+1)_{-1}^{j}(i+1)_{-1}^{s+j+1}(2i+1)_{-2}^{s+1}}{(j+1)!(2i+2)_{-2}^{j}},$$

$$\zeta_j^{2s+2,i+1} := \frac{2^{2j}(i+1)!(2s+2)!}{(s-j+1)!(2j)!(i-j-s)!}.$$

Proof. Due to the previous lemma, the actions of ad_x and ad_{N} on z^i are identical. Hence, our claim readily follows from Lemma 2.5. □

Now we define a vector space $\mathfrak{B}$ as

$$\mathfrak{B} := \mathrm{span}\{\mathfrak{b}_{0,0}^1 + \sum \beta_{i,k}^l \mathfrak{b}_{i,k}^l \,|\, -1 \leqslant l \leqslant 2i+1, i,k \in \mathbb{N}_0, \beta_{i,k}^l \in \mathbb{R}\}. \tag{3.4}$$

The following two lemmas show that $\mathfrak{B}$ is a Poisson algebra and it is Lie-isomorphic to $\mathscr{B}$.

Lemma 3.2. *The space $\mathfrak{B}$ is invariant under the Poisson bracket and the linear map*

$$\begin{aligned} \Psi : (\mathfrak{B}, \{\cdot,\cdot\} \to (\mathscr{B}, [\cdot,\cdot]), \\ \Psi(\mathfrak{b}_{i,k}^l) = \mathsf{B}_{i,k}^l, \end{aligned} \tag{3.5}$$

is a Lie isomorphism.

Proof. By the Leibniz rule we have

$$\begin{aligned} \mathrm{ad}_x^n(z^{i+1}) &= z\,\mathrm{ad}_x^n(z^i) + n\,\mathrm{ad}_x(z)\,\mathrm{ad}^{n-1}(x)(z^i) + \frac{n(n-1)}{2}\,\mathrm{ad}_x^2(z)\,\mathrm{ad}_x^{n-2}(z^i) \\ &= z\,\mathrm{ad}_x^n(z^i) + 2ny\,\mathrm{ad}_x^{n-1}(z^i) + n(n-1)x\,\mathrm{ad}_x^{n-2}(z^i). \end{aligned}$$

Due to Equation (3.5), we have $\Psi(x) = \mathrm{N}$, $\Psi(y) = \frac{\mathrm{H}}{2}$, and $\Psi(z) = -\mathrm{M}$. Further,

$$\mathrm{ad}_x^n(z^{i+1}\Delta^k) = \mathrm{ad}_x^n(z^{i+1})\Delta^k.$$

The actions of ad_x^n on $z^{i+1}\Delta^k$ and ad_{N} on $z^i\Delta^k\mathrm{M}$ are identified through Ψ. Then, the proof follows an induction on n, structure constants (3.1) and those governing the $\mathfrak{sl}_2$-triple M, N, and H. □

Now we present a ring structure constants for $\mathfrak{B}$ so that $\mathfrak{B}$ is a Poisson algebra.

Lemma 3.3. *The space $\mathfrak{B}$ is a Poisson algebra. In particular, let $q_1 = 2s_1 + r_1$ and $q_2 = 2s_2 + r_2$. Then, the ring structure constants are given by*

$$\begin{aligned} \mathfrak{b}_{i_1-1,k_1}^{q_1-1}\mathfrak{b}_{i_2-1,k_2}^{q_2-1} = \frac{-1}{i_1 i_2 \kappa_{q_1,2i_1}\kappa_{q_2,2i_2}} \\ \times \sum_{p=\max\{\sigma_2,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} \frac{\kappa_{2p+|r_2-r_1|,2(2p-\sigma_2+|r_2-r_1|)}C_{p,i,j}^{q_1,q_2}}{(2p-\sigma_2+|r_2-r_1|)^{-1}} \end{aligned}$$

$$\times\, \mathfrak{b}^{2p+|r_2-r_1|-1}_{2p-\sigma_2+|r_2-r_1|-1,\sigma_1-p+\lfloor\frac{r_1+r_2}{2}\rfloor}.$$

Proof. The proof directly follows from (3.3) and the formulas given in Theorem 2.7. Indeed, we have

$$\mathfrak{b}^{q_1-1}_{i_1-1,k_1}\mathfrak{b}^{q_2-1}_{i_2-1,k_2} = \frac{\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j)\Delta^{k_1+k_2}}{i_1 i_2 \kappa_{q_1,2i_1}\kappa_{q_2,2i_2}},$$

and

$$\frac{1}{i_1 i_2 \kappa_{q_1,2i_1}\kappa_{q_2,2i_2}} \sum_{p=\max\{\sigma_2,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} C^{q_1,q_2}_{p,i,j}\mathrm{N}^{2p+|r_2-r_1|}(z^{2p-\sigma_2+|r_2-r_1|})\Delta^{\sigma_1-p+\lfloor\frac{r_1+r_2}{2}\rfloor}$$

$$= \frac{-1}{i_1 i_2 \kappa_{q_1,2i_1}\kappa_{q_2,2i_2}}$$

$$\times \sum_{p=\max\{\sigma_2,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} \frac{\kappa_{2p+|r_2-r_1|,2(2p-\sigma_2+|r_2-r_1|)}C^{q_1,q_2}_{p,i,j}}{(2p-\sigma_2+|r_2-r_1|)^{-1}}\mathfrak{b}^{2p+|r_2-r_1|-1}_{2p-\sigma_2+|r_2-r_1|-1,\sigma_1-p+\lfloor\frac{r_1+r_2}{2}\rfloor}. \quad \square$$

The following theorem presents a property that is similar to the Hamiltonian cases, *i.e.,* the rate change of functions along with vector fields from $\mathscr{B}$ can be computed by the Poisson bracket.

Theorem 3.4. *For each l, i, k, we have*

$$\mathsf{B}^l_{i,k} = \sum_{j=1}^{3}\{x_j, \Psi^{-1}(\mathsf{B}^l_{i,k})\}\,\mathbf{e}_j, \qquad \textit{where} \qquad x_1 := x, x_2 := y, x_3 := z. \tag{3.6}$$

This representation for $B^l_{i,k}$ indicates that the family of vector fields from $\mathscr{B}$ and their associated dynamics are uniquely determined by their secondary Clebsch potentials. Furthermore, the change rate of any formal power series in (x, y, z), say $F : \mathbb{R}^3 \to \mathbb{R}$, along a vector field v from $\mathscr{B}$ is given by

$$\frac{dF}{dt} := \{F, \Psi^{-1}(v)\}. \tag{3.7}$$

Proof. From equations (2.12), (3.5) and (3.3), we have

$$\mathsf{B}^l_{i,k} = (l-2i-1)\mathfrak{b}^{l+1}_{i,k}\frac{\partial}{\partial x} + (l-i)\mathfrak{b}^l_{i,k}\frac{\partial}{\partial y} + (l+1)\mathfrak{b}^{l-1}_{i,k}\frac{\partial}{\partial z}.$$

Then, the proof follows the Lie isomorphism (3.5), the formulas (2.16)–(2.18), $\Psi(x) = \mathrm{N}$, $\Psi(y) = \frac{\mathrm{H}}{2}$, $\Psi(z) = -\mathrm{M}$, $\mathsf{B}^1_{0,0} = -\mathrm{N}$, and $\mathsf{B}^{-1}_{0,0} = -\mathrm{M}$. The vector field representation v from $\mathscr{B}$ in Poisson bracket form (3.7) directly follows from Equation (3.6), the linearity and the continuity (in filtration topology) of the Lie isomorphism ψ, the continuity and bilinearity of the Poisson bracket, and finally, the chain and Leibniz rules. $\square$

4. Geometrical features of integrable solenoidal vector fields

Definition 4.1.

- A vector field v is called *solenoidal* (nondissipative, incompressible, or volume-preserving) when $\operatorname{div}(v(x)) = 0$, and otherwise the vector field v is called *generalized dissipative*, *i.e.,* $\operatorname{div}(v(x)) \neq 0$.
- When $v(x) = \nabla f(x)$ for a scalar function $f(x)$, the vector field v is called a *gradient* or a *globally potential* vector field. Examples of this are the gravitational potential, a mechanical potential energy, and the electric potential energy.
- The vector field $v(x)$ is said to be nonpotential (non-gradient) when there exists at least a point $x \in \mathbb{R}^3$ such that $\mathbf{curl}(v(x)) \neq 0$; *e.g.,* see [43, page 1].

Theorem 4.2. *For every $v \in \mathscr{B}$, v is solenoidal.*

Proof. By the Leibniz rule and $\mathsf{B}^l_{i,k} = \Delta^k \mathsf{B}^l_{i,0}$, we observe that

$$\nabla \cdot \mathsf{B}^l_{i,k} = \nabla \cdot \Delta^k \mathsf{B}^l_{i,0} = \Delta^k \nabla \cdot \mathsf{B}^l_{i,0} + \nabla(\Delta^k) \cdot \mathsf{B}^l_{i,0}. \tag{4.1}$$

We claim that $\nabla \cdot \mathsf{B}^l_{i,0} = 0$ and $\nabla(\Delta^k) \cdot \mathsf{B}^l_{i,0} = 0$. Using Lemma 2.4, the partial derivatives of $\mathrm{N}^l(z^{i+1})$ are given by

$$\begin{aligned}
\frac{\partial}{\partial x}\mathrm{N}^{l+2}(z^{i+1}) &= (l+1)(l+2)(i+1)\mathrm{N}^l(z^i),\\
\frac{\partial}{\partial y}\mathrm{N}^{l+1}(z^{i+1}) &= 2(l+1)(i+1)\mathrm{N}^l(z^i),\\
\frac{\partial}{\partial z}\mathrm{N}^{l}(z^{i+1}) &= (i+1)\mathrm{N}^l(z^i).
\end{aligned} \tag{4.2}$$

Equation (2.12) and Lemma 2.4 give rise to

$$\nabla \cdot \mathsf{B}^l_{i,0} = (l+1)(i+1)\left(\frac{(l+2)(2i-l+1)}{\kappa_{l+2,2i+2}} + \frac{2(i-l)}{\kappa_{l+1,2i+2}} - \frac{1}{\kappa_{l,2i+2}}\right)\mathrm{N}^l(z^i) = 0. \tag{4.3}$$

On the other hand for $l = 2s$, Equation (2.12) gives rise to

$$\begin{aligned}
\nabla(\Delta^k) \cdot \mathsf{B}^l_{i,0} = {}& \frac{k(2i-s+1)z\Delta^{k-1}\mathrm{N}^{2s+2}(z^{i+1})}{(i+1)\kappa_{s+2,2i+2}} - \frac{2k(i-s)y\Delta^{k-1}\mathrm{N}^{2s+1}(z^{i+1})}{(i+1)\kappa_{s+1,2i+2}}\\
&- \frac{k(s+1)x\Delta^{k-1}\mathrm{N}^{2s}(z^{i+1})}{(i+1)\kappa_{s,2i+2}},
\end{aligned}$$

by applying Lemma 2.5 we obtain

$$2k\,i!(2s+1)!\sum_{n=0}^{s}\frac{\big((s+1)(i-n-s+1)-n(i-2s)-(i-s+1)(s-n+1)\big)x^{s-n+1}y^{2n}z^{i-n-s+1}\Delta^{k-1}}{2^{-2n}(2i-2s+1)!^{-1}(2n)!(s-n+1)!(i-n-s+1)!}.$$

Hence, $\nabla(\Delta^k)\cdot \mathsf{B}^l_{i,0}=0$ due to

$$(s+1)(i-j-s+1)-j(i-2s)-(i-s+1)(s-j+1)=0. \tag{4.4}$$

When $l=2s+1$, $\nabla(\Delta^k)\cdot \mathsf{B}^l_{i,0}$ is given by

$$\frac{2k(i-s)z\Delta^{k-1}\mathrm{N}^{2s+3}(z^{i+1})}{(i+1)\kappa_{2s+3,2i+2}}-\frac{2k(i-2s-1)y\Delta^{k-1}\mathrm{N}^{2s+2}(z^{i+1})}{(i+1)\kappa_{2s+2,2i+2}}$$
$$-\frac{2k(s+1)x\Delta^{k-1}\mathrm{N}^{2s+1}(z^{i+1})}{(i+1)\kappa_{2s+1,2i+2}},$$

and using Lemma 2.5 we find

$$=\sum_{n=0}^{s}\frac{2^{2n}(2s+1)!(i+1)!(2i-1-2s)!k}{(i-n-s-1)!(s-n)!(2i+2)!(2n)!(i+1)}$$
$$\left(\frac{4(i-s)(2s+2)_1^2}{(s+1-n)(2n+1)}-\frac{4(s+1)(2i-2s)_{-1-i}^2}{(s+1-n)(i-n-s)}-\frac{4(s+1)(2i-2s)_1^2}{(2n+1)(i-n-s)}\right)$$
$$\times x^{s+1-n}y^{2n+1}z^{i-n-s}\Delta^{k-1}$$
$$+\frac{ki!}{(2i+2)!}\left(\frac{(2i-2s)2^{2s+3}(2i-1-2s)!}{(i-2s-2)!}-\frac{(i-2s-1)2^{2s+3}(2i-2s)!}{(i-2s-1)!}\right)$$
$$\times y^{2s+3}z^{i-2s-1}\Delta^{k-1}.$$

Hence $\nabla(\Delta^k)\cdot \mathsf{B}^l_{i,0}=0$ due to the equation

$$\frac{(i-s)(2s+2)_1^2}{(s+1-j)(2j+1)}-\frac{(s+1)(2i-2s)_{-1-i}^2}{(s+1-j)(i-j-s)}-\frac{(s+1)(2i-2s)_1^2}{(2j+1)(i-j-s)}$$
$$=\frac{2(i-s)(2i-1-2s)!}{(i-2s-2)!}-\frac{(i-2s-1)(2i-2s)!}{(i-2s-1)!}=0. \tag{4.5}$$

Equations (4.5), (4.3), (4.4), and (4.1) conclude the proof. □

Theorem 4.3. *Let i and k be arbitrary nonnegative integers and $-1\le l\le i+1$.*

- *Polynomials $\mathfrak{b}^l_{i,0}$ and Δ are two first integrals for $\mathsf{B}^l_{i,k}$,* i.e.,

$$\mathsf{B}^l_{i,k}(\mathfrak{b}^l_{i,0})=0, \quad \textit{and} \quad \mathsf{B}^l_{i,k}(\Delta)=0.$$

Indeed for every $v\in\mathscr{B}$, $\Psi^{-1}(v)\in\mathfrak{B}$, and Δ are two first integrals for v.

- *A Clebsch potential representation for* $\mathsf{B}^l_{i,k}$ *is given by*

$$\mathsf{B}^l_{i,k} = \Delta^k(\nabla \mathfrak{b}^l_{i,0} \times \nabla \Delta). \tag{4.6}$$

  Equation (4.6) provides an alternative representation for each vector field v *in* $\mathscr{B}$ *by using the primary and secondary Clebsch potentials* Δ *and* $\Psi^{-1}(v) \in \mathfrak{B}$.
- *The polynomial functions* $\mathfrak{b}^l_{i,0}$ *and* Δ *are two functionally independent first integrals for* $\mathsf{B}^l_{i,0}$.
- *The ring of invariants for* $\mathsf{B}^{-1}_{i,k}$, $\mathsf{B}^i_{i,k}$, *and* $\mathsf{B}^{2i+1}_{i,k}$ *includes* $\langle \Delta, z\rangle$ (the algebra generated by Δ and z), $\langle \Delta, y\rangle$ and $\langle \Delta, x\rangle$, respectively.

Proof. By Equation (3.3), Leibniz rule and $\mathrm{N}z = \{x, z\} = 2y$, we have $\mathfrak{b}^l_{i,0} := -\frac{\mathrm{N}^{l+1}(z^{i+1})}{(i+1)\kappa_{l+1,2i+2}}$. Hence, Equation (2.12) and Lemma 2.4 imply

$$\begin{aligned}\mathsf{B}^l_{i,k}(\mathfrak{b}^l_{i,0}) = {} & \frac{(2i+1-l)!(l+1)\Delta^k}{-(i+1)\,(2i+2)!} \\ & \times\big(l\mathrm{N}^{l+2}(z^{i+1})\mathrm{N}^{l-1}(z^i) + 2(i-l)\mathrm{N}^{l+1}(z^{i+1})\mathrm{N}^l(z^i) \\ & - (2i+2-l)\mathrm{N}^l(z^{i+1})\mathrm{N}^{l+1}(z^i)\big).\end{aligned}$$

Now we claim that

$$\begin{aligned} & l\mathrm{N}^{l+2}(z^{i+1})\mathrm{N}^{l-1}(z^i) + 2(i-l)\mathrm{N}^{l+1}(z^{i+1})\mathrm{N}^l(z^i) \\ & \quad - (2i+2-l)\mathrm{N}^l(z^{i+1})\mathrm{N}^{l+1}(z^i) = 0. \end{aligned} \tag{4.7}$$

Equality (4.7) is trivial for the case $l = 0$. Let $l \neq 0$, $l := 2s + r$, $r = 0$ or 1. By Equation (2.7), we have

$$\mathsf{B}^l_{i,k}(\mathfrak{b}^l_{i,0}) = \sum_{n=0}^{l} f_r(n) x^{l-n} z^{2i-l-n} y^{2n+1},$$

where $f_r(n) := \sum_{p=0}^{n} F_r(n, p)$ for all $0 \leqslant n \leqslant l$. Now we follow Zeilberger's algorithm [35, Chapter 6] to prove $f_r(n) = 0$. By some computations one has

$$-2(2n+3)F_0(n+1, p) + (n-i-1)F_0(n, p) = G_0(n, p+1) - G_0(n, p), \tag{4.8}$$

where $G_0(n, p)$ and $F_0(n, p)$ are defined in Appendix C. Next, we add both sides of the equality (4.8) over p for all $0 \leqslant p \leqslant n-1$. Hence $G_0(n, n)$ is given by

$$\begin{aligned} & (n-i-1)f_0(n) - (4n+6)f_0(n+1) \\ & \quad + (4n+6)\big(F_0(n+1, n) + F_0(n+1, n+1)\big) - (n-i-1)F_0(n, n). \end{aligned}$$

On the other hand

$$G_0(n, n) = 2(2n+3)\big(F_0(n+1, n) + F_0(n+1, n+1)\big) - (n-i-1)F_0(n, n),$$

and

$$-2(2n+3)f_0(n+1)+(n-i-1)f_0(n)=0.$$

Thereby,

$$f_0(n)=\frac{f_0(0)}{2^n}\prod_{j=0}^{n-1}\frac{j-i-1}{2j+3},$$

also see [35, page 103]. Since

$$f_0(0)=F_0(0,p)=\frac{\frac{1}{i-s}+\frac{i-2s}{s(i-s)}-\frac{1}{s}}{(s-1)!s!\,(i-s)!\,(i-s-1)!}=0,$$

$f_0(n)=0$ for any n. Now let $l=2s+1$, *i.e.*, $r:=1$. Thus

$$-(3+2n)F_1(n+1,p)+2(n-i-1)F_1(n,p)=G_1(n,p+1)-G_1(n,p),$$

where $G_1(n,p)$ and $F_1(n,p)$ are given in Appendix C. Hence,

$$-(2n+3)f_1(n+1)+2(n-i-1)f_1(n)=0,$$

$f_1(0)=F_1(0,0)=0$ and finally, $f_1(n)=0$. Hence, $\mathsf{B}^l_{i,k}(\mathfrak{b}^l_{i,0})=0$ and $\mathfrak{b}^l_{i,0}$ is an invariant function for $\mathsf{B}^l_{i,k}$. Equation (4.1) and Theorem 4.2 give rise to $\mathsf{B}^l_{i,k}(\Delta)=\nabla(\Delta)\cdot\mathsf{B}^l_{i,k}=0$. Hence, Δ is also a first integral for $\mathsf{B}^l_{i,k}$.

By Lemma 2.4,

$$\nabla\mathfrak{b}^l_{i,0}=\frac{1}{\kappa_{l+1,2i+2}}\Big(l(l+1)\mathrm{N}^{l-1}(z^i),2(l+1)\mathrm{N}^l(z^i),\mathrm{N}^{l+1}(z^i))\Big).$$

Since $\mathsf{B}^l_{i,k}$ is tangent to the level surfaces of $\mathfrak{b}^l_{i,0}$ and Δ for any $(x,y,z)\in\mathbb{R}^3$, there exists a function $S^l_{i,k}(x,y,z)$ such that

$$\mathsf{B}^l_{i,k}=S^l_{i,k}\nabla\mathfrak{b}^l_{i,0}\times\nabla\Delta.$$

Hence,

$$(S^l_{i,0}\nabla\mathfrak{b}^l_{i,0}\times\nabla\Delta)\cdot\mathbf{e}_3-\mathsf{B}^l_{i,0}\cdot\mathbf{e}_3=-S^l_{i,0}\frac{ly\mathrm{N}^{l-1}(z^i)+z\mathrm{N}^l(z^i)}{\kappa_{l+1,2i+2}}+\frac{\mathrm{N}^l(z^{i+1})}{2(i+1)\kappa_{l,2i+2}}=0. \tag{4.9}$$

Therefore by Equation (2.5),

$$\frac{\sum_{n=0}^{s}\zeta_n^{2s,i+1}x^{s-n}y^{2n}z^{i-n-s+1}}{2(i+1)\kappa_{l,2i+2}}-\frac{lS_{i,0}^{l}\sum_{n=1}^{s}\zeta_{n-1}^{2s-1,i}x^{s-n}y^{2n}z^{i-n-s+1}}{\kappa_{l+1,2i+2}}$$
$$-\frac{S_{i,0}^{l}\sum_{n=0}^{s}\zeta_n^{2s,i}x^{s-n}y^{2n}z^{i-n-s+1}}{\kappa_{l+1,2i+2}}=0.$$

Thus, $S_{i,0}^{l}=1$ due to

$$\frac{l\zeta_{n-1}^{2s-1,i}+\zeta_n^{2s,i}}{\kappa_{l+1,2i+2}}-\frac{\zeta_n^{2s,i+1}}{2(i+1)\kappa_{l,2i+2}}=0, \tag{4.10}$$

for all $0\leqslant n\leqslant s$, and $\zeta_{-1}^{2s-1,i}=0$. The condition $S_{i,0}^{l}=1$ implies $S_{i,k}^{l}=\Delta^k$.

Since $\mathsf{B}_{i,0}^{l}\neq 0$ for almost everywhere (except for a set with zero Lebesgue measure), the last two claims are immediately concluded from the first and second claim, and Lemma 4.4. □

Define the grading function

$$\delta(\mathsf{B}_{i,k}^{l}):=i+2k, \tag{4.11}$$

that is, the standard degree of homogeneous vector fields minus one. This makes the space $(\mathscr{B},[-,-])$ a graded Lie algebra. Hence, for $N\in\mathbb{N}_0$ the vector space

$$\mathscr{B}_N:=\operatorname{span}\{\mathsf{B}_{N-2k,k}^{l}:l=-1,\dots,2(N-2k)+1,k=0,\dots,\lfloor\frac{N}{2}\rfloor\},$$

consists of all δ-homogenous vector fields of grade N. For $v\in\mathscr{B}$, we define (also see [14, Definition 3.2]) $\operatorname{Terms}(v):=\cup_{p=1}^{3}\operatorname{Terms}(v\cdot\mathbf{e}_p)$, while

$$\operatorname{Terms}(v\cdot\mathbf{e}_p):=\{\text{All monomials contributing in Taylor expansion } v\cdot\mathbf{e}_p\}.$$

For an instance we have $\operatorname{Terms}(2x^2+3xy+5)=\{x^2,xy,1\}$. Thereby, for any two δ-grade homogeneous vector fields v_1 and v_2, where $\delta(v_1)\neq\delta(v_2)$,

$$\operatorname{Terms}(v_1\cdot\mathbf{e}_p)\cap\operatorname{Terms}(v_2\cdot\mathbf{e}_p)=\varnothing,\ \text{ for any } p=1,2,3.$$

When $i,k\in\mathbb{N}_0$, $-1\leqslant l\leqslant 2i+1$, and $N:=i+2k$, we define a condition for a nonnegative integer m by

$$0\leqslant m\leqslant\min\left\{\lfloor\frac{N}{2}\rfloor,\lfloor\frac{2N-2k-l+1}{2}\rfloor,\lfloor\frac{2k+l+1}{2}\rfloor\right\}, \tag{4.12}$$

and next, a set $P_{i,k}^{l}$ by

$$P_{i,k}^{l}:=\{(m_1,m_2,m_3):m_1=l+2(k-m_3),m_2=N-2m_3,\text{the condition (4.12) holds for } m_3\}.$$

Lemma 4.4. *Let* $\mathsf{B}^l_{i,k} \in \mathscr{B}$, $(m_1, m_2, m_3) \in P^l_{i,k}$, *and* $p \in \{1, 2, 3\}$. *Then,*

1. $\mathrm{Terms}(\mathsf{B}^{m_1}_{m_2,m_3} \cdot \mathbf{e}_p) \neq \varnothing$ *when* $l \neq (3-p)i + 2 - p$. *Otherwise,* $\mathrm{Terms}(\mathsf{B}^{m_1}_{m_2,m_3} \cdot \mathbf{e}_p) = \varnothing$, i.e., *for* $(p, l) = (1, 2i+1)$, $(p, l) = (2, i)$ *and* $(p, l) = (3, -1)$.
2. *When* $\mathrm{Terms}(\mathsf{B}^l_{i,k} \cdot \mathbf{e}_p) \neq \varnothing$,

$$\mathrm{Terms}(\mathsf{B}^{m_1}_{m_2,m_3} \cdot \mathbf{e}_p) \subseteq \mathrm{Terms}(\mathsf{B}^l_{i,k} \cdot \mathbf{e}_p) \text{ and } P^l_{i,k} = P^{m_1}_{m_2,m_3}.$$

3. *Let* $\mathrm{Terms}(\mathsf{B}^l_{i,k}) = \mathrm{Terms}(\mathsf{B}^{l'}_{i',k'})$. *Then, natural numbers* $l - l'$ *and* $i - i'$ *are even. Furthermore, the inequalities* $k \neq k', l \neq l'$ *and* $i \neq i'$ *are equivalent. In particular,* $k = k'$ *implies* $(l, i, k) = (l', i', k')$.
4. *The set*

$$\{\mathrm{Terms}(\mathsf{B}^{-1}_{N,0} \cdot \mathbf{e}_p), \mathrm{Terms}(\mathsf{B}^{N}_{N,0} \cdot \mathbf{e}_p), \mathrm{Terms}(\mathsf{B}^{2N+1}_{N,0} \cdot \mathbf{e}_p), \mathrm{Terms}(\mathsf{B}^{l}_{N,0}) \mid 0 \leq l \leq 2N, l \neq N\},$$

is a partition for

$$\cup\{\mathrm{Terms}(v_N \cdot \mathbf{e}_p), v_N \in \mathscr{B}\}.$$

5. $\mathrm{Terms}(\mathfrak{b}^l_{i,k}) = \mathrm{Terms}(\mathfrak{b}^{m_1}_{m_2,m_3}) \neq \varnothing$.
6. *For any* $0 \neq v \in B$ *and nonnegative integer* N, *there exists a unique polynomial vector*

$$v^j_{N-2k,0} \in \mathbb{R}\{\mathsf{B}^j_{N-2k,0}\},$$

for each $-1 \leqslant j \leqslant 2N - 4k + 1$ *and* $0 \leqslant k \leqslant \min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lceil \frac{N}{2} \rceil\}$ *so that*

$$v = \sum_{N=0}^{\infty} \sum_{j=-1}^{2N+1} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} v^j_{N-2k,0} \Delta^k. \tag{4.13}$$

Furthermore, $v = 0$ *if and only if* $v^j_{N-2k,0} = 0$ *for all* j, k, N.

Proof. The claim in part 1 directly follows from Equation (2.12). For claim 2, let $\mathsf{B}^{l'}_{i',k'}$ be the vector field in $\mathscr{B}$ with $(l, i, k) \neq (l', i', k')$. Here we only consider the case $p = 3$. Using the polynomial expansion for Δ^k, $\Delta^{k'}$, Lemma 2.5, and Definition (2.11), the monomials appearing in $\mathsf{B}^l_{i,k}$ and $\mathsf{B}^{l'}_{i',k'}$ follow

$$P(x, y, z) = x^{s-n_1+p_1} y^{2n_1+r+2(k-p_1)} z^{i-n_1-s-r+p_1},$$
$$Q(x, y, z) = x^{s'-n_2+p_2} y^{2n_2+r'+2(k'-p_2)} z^{i'-n_2-s'-r'+p_2},$$

for some n_1, n_2, p_1, p_2, where $l = 2s + r$ and $l' = 2s' + r'$. Let $P = Q$. Then,

$$\begin{aligned} s - s' &= n_1 - n_2 + p_2 - p_1, \\ r - r' &= 2(n_2 - n_1 + k' - k + p_1 - p_2), \\ i - i' &= n_1 - n_2 + s - s' + r - r' + p_2 - p_1. \end{aligned} \tag{4.14}$$

By substituting the first equation in (4.14) into the third one, we have

$$i - i' = l - l'. \tag{4.15}$$

Since the vector fields $\mathsf{B}^l_{i,k}$ and $\mathsf{B}^{l'}_{i',k'}$ have the same δ-grade,

$$i - i' = 2(k' - k). \tag{4.16}$$

Therefore, $i - i' = l - l' = 2(k' - k)$ and $i + 2k = i' + 2k'$. Hence, the inequalities $i' \geq 0$ and $-1 \leq l' \leq 2i' + 1$ are equivalent to the inequalities (4.12) on $m := k'$. The equality $P^l_{i,k} = P^{m_1}_{m_2,m_3}$ is due to the definition.

Part 3 follows equations (4.16) and (4.15). The claim 4 is a direct corollary of the claim 2. The proof of part 5 is similar to claims 2–4 and the claim is consistent with Equation (4.6).

Finally for the claim 6, consider $v := \sum_{N=0}^{\infty} w_N$ for $w_N \in \mathscr{B}_N$. Then by claim 5, for any N we have

$$w_N := \sum_{j=-1}^{2N+1} w^j_N, \qquad w^j_N \cdot \mathbf{e}_p \in \operatorname{span} \operatorname{Terms}(\mathsf{B}^j_{N,0} \cdot \mathbf{e}_p).$$

Now we Taylor-expand w^j_N in terms of Δ, that is,

$$\begin{aligned} w^j_N &= \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} v^j_{N-2k,0} \Delta^k, \\ &\text{where} \quad v^j_{N-2k,0} \cdot \mathbf{e}_p \in \operatorname{span} \operatorname{Terms}(\mathsf{B}^j_{N-2k,0} \cdot \mathbf{e}_p). \end{aligned} \tag{4.17}$$

Hence, $v^j_{N-2k,0}$ does not share any monomial vector field with any B-terms except $\mathsf{B}^j_{N-2k,0}$. This is because of the claim in part 2 and that we have already excluded the powers of Δ in (4.17). Therefore, the expansion of $v^j_{N-2k,0}$ in terms of the B-term generators of $\mathscr{B}$ only includes $\mathsf{B}^j_{N-2k,0}$. $\square$

The following definitions (for the spaces $\mathscr{C}$ and $\mathscr{A}$) and its subsequent two theorems describe two families of $\mathfrak{sl}_2$-invariant vector fields. These two families provide a decomposition for all three dimensional vector fields for normal form derivation of three dimensional nilpotent singularity. However, further study of these two families are beyond the scope of this paper. Let

$$\mathscr{C} := \operatorname{span} \left\{ \sum c^l_{i,k} \mathsf{C}^l_{i,k} \,\middle|\, -2 \leqslant l \leqslant 2i + 2, i, k \in \mathbb{N}_0, c^l_{i,k} \in \mathbb{R} \right\}, \tag{4.18}$$

and

$$\mathscr{A} := \operatorname{span}\left\{\sum a^l_{i,k}\mathsf{A}^l_{i,k} \,\middle|\, 0 \leqslant l \leqslant 2i, i, k \in \mathbb{N}_0, a^l_{i,k} \in \mathbb{R}\right\}, \tag{4.19}$$

where

$$\mathsf{C}^l_{i,k} := \frac{1}{\kappa_{l+2,2i+2}}\mathrm{N}^{l+2}(z^i\Delta^k\frac{\partial}{\partial x}), \qquad \text{for } -2 \le l \le 2i+2, i, k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \tag{4.20}$$

$$\mathsf{A}^l_{i,k} := \frac{1}{\kappa_{l,2i}}\mathrm{N}^l(z^i\Delta^k\mathrm{E}), \qquad \text{for } \quad 0 \le l \le 2i, i, k \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}. \tag{4.21}$$

By Equation (2.4), we have

$$\begin{aligned}
\mathsf{C}^l_{i,k} &:= \frac{\Delta^k\mathrm{N}^{l+2}(z^{i+1})}{\kappa_{l+2,2i+2}}\frac{\partial}{\partial x} - \frac{(l+2)\Delta^k\mathrm{N}^{l+1}(z^{i+1})}{(2i-l+1)\kappa_{l+1,2i+2}}\frac{\partial}{\partial y} + \frac{(l+1)(l+2)\Delta^k\mathrm{N}^l(z^{i+1})}{(2i-l+1)(2i-l+2)\kappa_{l,2i+2}}\frac{\partial}{\partial z},\\
\mathsf{A}^l_{i,k} &:= \frac{x\Delta^k\mathrm{N}^l(z^i)}{\kappa_{l,2i}}\frac{\partial}{\partial x} + \frac{y\Delta^k\mathrm{N}^l(z^i)}{\kappa_{l,2i}}\frac{\partial}{\partial y} + \frac{z\Delta^k\mathrm{N}^l(z^i)}{\kappa_{l,2i}}\frac{\partial}{\partial z}.
\end{aligned} \tag{4.22}$$

Theorem 4.5. *For all* $-2 \leqslant l \leqslant 2i+2, i, k \in \mathbb{N}_0$,

$$\operatorname{div}(\mathsf{C}^l_{i,0}) = 0, \qquad \nabla\Delta\cdot\mathsf{C}^l_{i,k} \neq 0. \tag{4.23}$$

Proof. By Equation (4.2) and definition $\mathsf{C}^l_{i,k}$, we have

$$\begin{aligned}
\nabla\cdot\mathsf{C}^l_{i,0} = {} & (l+1)(l+2)(i+1)\\
& \times\Big(\frac{1}{\kappa_{l+2,2i+2}} - \frac{2}{\kappa_{l+1,2i+2}(2i-l+1)} - \frac{1}{\kappa_{l,2i+2}(2i-l+1)(2i-l+2)}\Big)\mathrm{N}^l(z^i).
\end{aligned}$$

Since the coefficient $\mathrm{N}^l(z^i)$ is zero, $\operatorname{div}(\mathsf{C}^l_{i,k}) = 0$ for any $i \in \mathbb{N}_0$, and $-2 \leqslant l \leqslant 2i+2$.

When l is even, say $l = 2s$, by Lemma 2.5 we have

$$\begin{aligned}
\nabla\Delta\cdot\mathsf{C}^l_{i,0} = {} & \frac{(2s+2)!(i+1)!(2i-2s)!}{(2i+2)!}\sum_{n=0}^{s+1}\frac{4^n x^{s-n+1}y^{2n}z^{i-n-s+1}}{(s-n+1)!(2n)!(i-n-s)!}\\
& + \frac{(2s+2)!(i+1)!(2i-2s)!}{(2i+2)!}\sum_{n=0}^{s}\frac{4^n x^{s-n+1}y^{2n}z^{i-n-s+1}}{(s-n)!(2n)!(i-n-s+1)!}\\
& + \frac{4(2i-2s)!(s+1)}{(2i+2)!}\sum_{n=1}^{s+1}\frac{(2s+1)!(i+1)!2^{2n-1}x^{s-n+1}y^{2n}z^{i-n-s+1}}{(s-n+1)!(2n-1)!(i-n-s+1)!}.
\end{aligned}$$

Since the coefficient of $x^{s+1}z^{i-s+1}$ is

$$\frac{(i+2)!\,(2i-2s)!\,(2s+2)!}{(2i+2)!\,(s+1)!\,(i-s+1)!} \neq 0,$$

Δ is not a first integral for $\mathsf{C}^l_{i,k}$. The argument is similar for when l is odd. $\square$

Theorem 4.6. *Each three dimensional vector field v can be uniquely expanded in terms of formal sums of polynomial generators $\mathsf{C}^l_{i,k}$, $\mathsf{B}^l_{i,k}$ and $\mathsf{A}^l_{i,k}$ from $\mathscr{C}$, $\mathscr{B}$ and $\mathscr{A}$, respectively.*

Proof. This is a straightforward corollary of Theorem 2.9, Lemma 4.4, and Remark 4.7. □

Remark 4.7. Given Lemma 4.4, the same statements trivially hold for other $\mathfrak{sl}_2$-generated vector fields from spaces $\mathscr{C}$ and $\mathscr{A}$. In particular, for any $v \in \mathscr{A}$ and $w \in \mathscr{C}$, there exist uniquely determined constants $c^j_{N,k}$ and $a^j_{N,k}$ so that

$$v = \sum_{N=0}^{\infty} \sum_{j=0}^{2N} \sum_{k=0}^{\min\{\lfloor \frac{2N-j}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} a^j_{N,k} \mathsf{A}^j_{N-2k,0} \Delta^k, \tag{4.24}$$

$$w = \sum_{N=0}^{\infty} \sum_{j=-2}^{2N+2} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+2}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} c^j_{N,k} \mathsf{C}^j_{N-2k,0} \Delta^k. \tag{4.25}$$

The following theorem provides a concrete characterization for vector fields in $\mathscr{B}$.

Theorem 4.8. *Let v be a three dimensional vector field. The conditions* $\operatorname{div}(v) = 0$ *and* $v(\Delta) = 0$ *hold if and only if $v \in \mathscr{B}$.*

Proof. Given Theorem 4.2 and Theorem 4.3 (first item), we only need to prove the *only if* part. Let $v = u + w_C + w_A$, where $u \in \mathscr{B}$, $w_A \in \mathscr{C}$, and $w_C \in \mathscr{A}$. Hence, $\nabla \cdot (w_C + w_A) = 0$, and $w_C(\Delta) + w_A(\Delta)$. By equations (4.24) and (4.25),

$$w_C + w_A = \sum_{N=0}^{\infty} \Big(\sum_{j=0}^{2N} \sum_{k=0}^{\min\{\lfloor \frac{2N-j}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} {w_C}^j_{N-2k,0} \Delta^k + \sum_{j=-2}^{2N+2} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+2}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} {w_A}^j_{N-2k,0} \Delta^k \Big),$$

where ${w_C}^j_{N-2k,0} \in \mathbb{R}\{\mathsf{A}^j_{N-2k,0}\}$ and ${w_A}^j_{N-2k,0} \in \mathbb{R}\{\mathsf{C}^j_{N-2k,0}\}$. Since

$$\operatorname{Terms}((\mathbb{R}\{\mathsf{A}^{j_1}_{N-2k_1,0}\} + \mathbb{R}\{\mathsf{C}^{j_1}_{N-2k_1,0}\}) \cdot \mathbf{e}_p) \cap \operatorname{Terms}((\mathbb{R}\{\mathsf{A}^{j_2}_{N-2k_2,0}\} + \mathbb{R}\{\mathsf{C}^{j_2}_{N-2k_2,0}\}) \cdot \mathbf{e}_p) = \varnothing$$

for all $p = 1, 2, 3$ when $(j_1, k_1) \neq (j_2, k_2)$,

$$\operatorname{div} {w_C}^j_{N-2k,0} + \operatorname{div} {w_A}^j_{N-2k,0} = 0 \quad \text{and} \quad {w_C}^j_{N-2k,0}(\Delta) + {w_A}^j_{N-2k,0}(\Delta) = 0.$$

By Theorem 4.5, $\operatorname{div} {w_A}^j_{N-2k,0} = 0$ and so $\operatorname{div} {w_C}^j_{N-2k,0} = 0$ for all j, k, N. By Lemma 2.4, Equation (4.22), and item 6 in Lemma 4.4, the condition $\operatorname{div} {w_C}^j_{N-2k,0} = 0$ implies that ${w_C}^j_{N-2k,0} = 0$. Thereby, ${w_A}^j_{N-2k,0}(\Delta) = 0$. Now Theorem 4.5 concludes the proof through the equation ${w_A}^j_{N-2k,0} = 0$ for all j, k, N. Indeed, $w_A = w_C = 0$ and $v = u \in \mathscr{B}$. □

Theorem 4.9. *The following hold.*

1. *The set* $\{\mathrm{Terms}(\mathfrak{b}^l_{N,0}) : -1 \leqslant l \leqslant 2N+1\}$ *forms a partition for* $\mathfrak{B}_N = \Psi^{-1}(\mathscr{B}_N)$.
2. *For* $p = 1, 2, 3$, *and* $(x_1, x_2, x_3) := (x, y, z)$,

$$\{x_p, \mathrm{Terms}(\mathfrak{b}^l_{i,k})\} \subseteq \mathrm{Terms}(\mathfrak{b}^{l+2-p}_{i,k}).$$

When $(p, l) \neq (1, 2i+1)$, $(p, l) \neq (2, i)$ *and* $(p, l) \neq (3, -1)$,

$$\mathrm{Terms}\{x_p, \mathrm{Terms}(\mathfrak{b}^l_{i,k})\} = \mathrm{Terms}(\mathfrak{b}^{l+2-p}_{i,k}).$$

3. $\ker \mathrm{ad}_x = \mathbb{R}[[x]]$, $\ker \mathrm{ad}_y = \mathbb{R}[[y, xz]]$, *and* $\ker \mathrm{ad}_z = \mathbb{R}[[z]]$.

Proof. The first item follows items 3 and 4 in Theorem 4.4, and Lemma 3.2. The second and third claim follows from the structure constants and Lemma 3.2. □

An alternative representation for vector fields in $\mathscr{B}$ are based on the vector potential. Each solenoidal vector field v has always a vector potential that is unique modulo gradient vector fields. Vector potential frequently appears in the classical and quantum mechanics, *e.g.,* see [25]. Vector potential is called magnetic vector potential in electrodynamics while the curl of the magnetic vector potential is called magnetic field; see [4].

Theorem 4.10. *There exists a vector potential* $\phi^l_{i,k}$ *such that* $\mathsf{B}^l_{i,k} = \mathbf{curl}(\phi^l_{i,k})$, *where*

$$\phi^l_{i,k} := \mathfrak{b}^l_{i,k} \nabla \Delta = \frac{\Delta^k \mathrm{N}^{l+1}(z^{i+1})}{\kappa_{l+1,2i+2}} \big(z, -2y, x\big). \tag{4.26}$$

Proof. From Equation (4.6) and the equality $\nabla f \times \nabla g = \nabla \times f \nabla g$ for all scalar functions f and g, we have

$$\mathsf{B}^l_{i,k} = \nabla \times (\mathfrak{b}^l_{i,k} \nabla \Delta).$$

Thereby, $\mathfrak{b}^l_{i,k} \nabla \Delta$ is a vector potential for $\mathsf{B}^l_{i,k}$. □

Remark 4.11. An alternative vector potential for solenoidal vector fields is available through the computational approach on [26, page 21]. Indeed, there exists a vector potential $\Phi^l_{i,k}$ such that $\mathsf{B}^l_{i,k} = \mathbf{curl}(\Phi^l_{i,k})$, where

$$\begin{aligned}\Phi^l_{i,k} = \frac{\Delta^k}{(i+1)(2k+i+3)} \times \\ \Big((l-i) z \mathrm{N}^{l+1}(z^{i+1}) - (l+1) y \mathrm{N}^l(z^{i+1}) \cdot \mathbf{e}_1 \\ + (2i+1-l) z \mathrm{N}^{l+2}(z^{i+1}) + (l+1) x \mathrm{N}^l(z^{i+1}) \cdot \mathbf{e}_2 \\ (l-2i-1) y \mathrm{N}^{l+2}(z^{i+1}) + (i-l) x \mathrm{N}^{l+2}(z^{i+1}) \cdot \mathbf{e}_3\Big).\end{aligned} \tag{4.27}$$

In particular, $\Phi^1_{0,0} = (-\Delta, 0, 0)$. The proof here follows [26, page 21]. Indeed, define

$$P(X) := \int_0^1 t\, \mathsf{B}^l_{i,k}(tX)\, dt = \frac{1}{(2k+i+3)} \mathsf{B}^l_{i,k},$$

where $X := (x, y, z)$. Then, a vector potential for $\mathsf{B}^l_{i,k}$ is given by $P(X) \times X$. Hence, Equation (4.27) is computed through Equation (2.12).

We further recall that a vector potential for a given solenoidal vector field is generally unique modulo gradient vector fields. For instance by equations (4.26) and (4.27), vector fields

$$\phi^1_{1,0} = \left(\frac{y^2 z}{4}, -\frac{xyz}{2}, \frac{xy^2}{4} \right),$$
$$\Phi^1_{1,0} = \frac{1}{3}\left(xz + 2y^2\right)\left(\frac{1}{2}z, -y, \frac{1}{2}x\right),$$

are two different vector potentials for $\mathsf{B}^1_{1,0}$. Here, $\phi^1_{1,0} + \nabla f = \Phi^1_{1,0}$ where $f = \frac{1}{12}zxy^2 - \frac{1}{6}y^4 + \frac{1}{12}x^2z^2$.

Nonlinear vector fields from $\mathscr{B}$ are rotational vector fields; *i.e.,* they have a nonzero curl.

Theorem 4.12. *All vector fields from $\mathscr{B}$ have a non-zero curl. In particular, the null space of curl operator on the formal sum of vector field types* (2.11) *is given by* $\mathbb{R}\mathsf{B}^0_{0,0}$.

Proof. Note that $\mathbf{curl}(\mathsf{B}^0_{0,0}) = 0$. Let $v \in \mathscr{B}$ and $\mathbf{curl}(v) = 0$. By Equation (4.13), Lemma 4.4, linearity and continuity in filtration topology of **curl** operator,

$$v = \sum_{N=0}^{\infty} \sum_{j=-1}^{2N+1} \sum_{k=0}^{\min\{\lfloor \frac{2N-j+1}{4} \rfloor, \lceil \frac{N}{2} \rceil\}} v^j_{N-2k,0} \Delta^k, \quad \text{where} \quad v^j_{N-2k,0} \in \mathbb{R}\{\mathsf{B}^j_{N-2k,0}\},$$

and $\mathbf{curl}(v^j_{N-2k,0}\Delta^k) = 0$ for all j, l, N. Now let $\mathbf{curl}(\mathsf{B}^l_{i,k}) = 0$. Hence, all three components of $\mathbf{curl}(\mathsf{B}^l_{i,k})$ are zero. The first component of $\mathbf{curl}(\mathsf{B}^l_{i,k})$ is given by

$$-\frac{\partial}{\partial y} \frac{(l+1)\Delta^k \mathrm{N}^l(z^{i+1})}{(i+1)\kappa_{l,2i+2}} - \frac{\partial}{\partial z} \frac{(i-l)\Delta^k \mathrm{N}^{l+1}(z^{i+1})}{(i+1)\kappa_{l+1,2i+2}},$$

and by Lemma 2.4, we have

$$\nabla \times \mathsf{B}^l_{i,k} \cdot \mathbf{e}_1 = -\frac{2l(l+1)\Delta^k \mathrm{N}^{l-1}(z^i)}{\kappa_{l,2i+2}} + \frac{2yk(l+1)\Delta^{k-1}\mathrm{N}^l(z^{i+1})}{(i+1)\kappa_{l,2i+2}} - \frac{(i-l)\Delta^k \mathrm{N}^{l+1}(z^i)}{\kappa_{l+1,2i+2}}$$
$$- \frac{k(i-l)x\Delta^{k-1}\mathrm{N}^{l+1}(z^{i+1})}{(i+1)\kappa_{l+1,2i+2}} = 0. \tag{4.28}$$

Since $\mathrm{Terms}(\Delta^k \mathrm{N}^{l-1}(z^i))$, $\mathrm{Terms}(\Delta^{k-1}\mathrm{N}^l(z^{i+1}))$, $\mathrm{Terms}(\Delta^k \mathrm{N}^{l+1}(z^i))$, and $\mathrm{Terms}(\mathrm{N}^{l+1}(z^{i+1}))$ are pairwise disjoint sets of monomial terms, Equation (4.28) holds if and only if

$$l(l+1) = k(l+1) = (i-l) = k(i-l) = 0.$$

The later is equivalent with $i = k = l = 0$. This completes the proof. □

Example 4.13. Let

$$v := \mathsf{C}^0_{0,1} - 3\mathsf{A}^2_{2,0} = -3xy^2\frac{\partial}{\partial x} - 3xyz\frac{\partial}{\partial y} - 3y^2z\frac{\partial}{\partial z}.$$

Then, $v(\Delta) = 0$ while $\mathrm{div}(v) = -3xz - 6y^2 \neq 0$.

Consider the vector field

$$\mathsf{C}^{-2}_{i,0} := z^{i+1}\frac{\partial}{\partial x}, \qquad \text{for any } i \in \mathbb{N}_0.$$

This family has two first integrals of y and z while $\mathsf{C}^{-2}_{i,0}$ is also solenoidal for all i. These vector fields do not generate a Lie algebra with the nilpotent linear part $\mathsf{B}^1_{0,0}$, indeed,

$$[\mathsf{C}^{-2}_{j,0}, \mathsf{C}^{-2}_{i,0}] = 0, \qquad [\mathsf{B}^1_{0,0}, \mathsf{C}^{-2}_{i,0}] = -2(i+1)\mathsf{C}^{-1}_{i,0}.$$

This indicates that the family of vector fields in $\mathscr{B}$ does not represent the set of all solenoidal vector fields with two independent first integrals.

5. Normal form classification

This section is devoted to obtain the normal forms of the vector fields from and within the Lie algebra $\mathscr{B}$. In other words, the normal form vector field of a vector field from $\mathscr{B}$ remains a completely integrable solenoidal vector field, where Δ is one of its first integrals. Alternative normal form vector field representations such as vector potential and the Clebsch potential normal form are also provided.

Theorem 5.1. *The vector field* (1.1)–(1.2) *is either linearizable in the normalization process or there exists a natural number p so that the normal form of the vector field* (1.1)–(1.2) *is given by*

$$\begin{aligned}\mathbf{w} := &-x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + z^p\left(z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}\right)\\ &+ \sum_{k=p+1}^{\infty}\sum_{i=0}^{[\frac{k+p}{2}]} b_{i,k} z^i (xz - y^2)^{k-2i+p}\left(z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}\right),\end{aligned} \tag{5.1}$$

where $b_{i,k} \in \mathbb{R}$. Furthermore, the normal form vector field (5.1) *can not be further simplified, that is, the normal form coefficients are uniquely determined in term of the original system* (1.1)–(1.2). *In addition, the secondary Clebsch potential normal form is given by*

$$\mathbf{I}(x,y,z) = x + \frac{1}{p+1}z^{p+1} + \sum_{k=p+1}^{\infty}\sum_{i=0}^{[\frac{k+p}{2}]}\frac{b_{i,k}}{i+1}z^{i+1}(xz-y^2)^{k-2i+p}. \tag{5.2}$$

Here, the polynomial $\Delta = xz - y^2$ stands for the primary Clebsch potential.

Proof. The normal form

$$\mathbf{w} := \mathsf{B}^1_{0,0} + \sum_{i,k\in\mathbb{N}_0}^{\infty} b_{i,k}\mathsf{B}^{-1}_{i,k}, \tag{5.3}$$

is readily available given the $\mathfrak{sl}_2$-style normal form and the fact that $\ker(\mathrm{ad}_{\mathrm{M}}) = \mathrm{span}\{\mathsf{B}^{-1}_{i,k}\}$; for more information see [2,3]. Let $p := \min\{i \mid b_{i,0} \neq 0\}$ and $p > 0$. Define a new grading function by $\delta(\mathsf{B}^l_{i,k}) := lp + 2i + k$. Then, $\mathscr{B}$ is a δ-graded Lie algebra and

$$\mathbb{B}_p := \mathsf{B}^1_{0,0} + b_{p,0}\mathsf{B}^{-1}_{p,0} \in \mathscr{B}_p.$$

Linear invertible transformations can be used to rescale the coefficient $b_{p,0}$ into $b_{p,0} := 1$. Following [2,3] we define

$$\Gamma := \mathrm{ad}(\mathsf{B}^{-1}_{0,0}) \circ \mathrm{ad}(\mathbb{B}_p). \tag{5.4}$$

By the structure constants and Equation (2.19),

$$\begin{aligned}\Gamma(\mathsf{B}^q_{i,k}) &= (q-2i-1)(q+2)\mathsf{B}^q_{i,k} \\ &\quad + b_{p,0}\sum_{j=\max\{q-2-i-p,-1\}}^{s-1+\lfloor\frac{r+1}{2}\rfloor}\frac{a^{-1,q}_{j,p,i}\mathsf{B}^{2j+|r-1|-1}_{2j-q+1+i+p+|r-1|,s-1+k-j+\lfloor\frac{r+1}{2}\rfloor}}{(2j+|r-1|+1)^{-1}}.\end{aligned}$$

Hence for any i and k when $q = 2i+1$, there is a possibility of a vector polynomial in kernel Γ. This is due to a similar argument used by [2,3]. On the other hand

$$\mathfrak{b}^{2i+1}_{i,k} = -x^{i+1}\Delta^k,$$

and

$$\Psi((\mathfrak{b}^1_{0,0})^{i+1}\Delta^k) = \mathsf{B}^{2i+1}_{i,k},$$

in addition,

$$\mathbb{B}^{i+1}_r\Delta^k := \Psi\left((\Psi^{-1}(\mathbb{B}_r))^{i+1}\Delta^k\right) = \Psi((-x-z^{r+1})^{i+1}\Delta^k) \in \ker\Gamma.$$

These polynomial vectors are extended to a symmetry for the normal form vector field (5.1), through

$$\Psi\left((\Psi^{-1}(\mathbf{w})^{i+1}\Delta^k\right).$$

This proves that there is no possibility of any hypernormalization beyond the $\mathfrak{sl}_2$-style normalized vector field (5.1). □

Corollary 5.2. *The following presents five alternative representations for the normal form* (5.1)*:*

1. *A formal sum of* B*-terms:*

$$\mathbf{w} := \mathsf{B}^1_{0,0} + \mathsf{B}^{-1}_{p,0} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{[\frac{k+p}{2}]} b_{i,k} \mathsf{B}^{-1}_{i,k+p-2i}.$$

2. *The secondary invariant:*

$$\mathbf{w} := -\Psi\Big(x + z^p + \sum_{k=p+1}^{\infty} \sum_{i=0}^{[\frac{k+p}{2}]} b_{i,k} z^{i+1} (xz - y^2)^{k+p-2i}\Big).$$

   Here, Ψ *is the Lie isomorphism given by Equation* (3.5)*.*

3. *Vector potential:*

$$\mathbf{w} := \mathbf{curl}\Big((x + z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{[\frac{k+p}{2}]} \frac{\Delta^k z^{i+1}}{i+1})(z, -2y, x)\Big).$$

4. *Functionally independent Clebsch potentials:*

$$\mathbf{w} := \nabla(xz - y^2) \times \nabla\Big(x + z^{p+1} + \sum_{k=p+1}^{\infty} \sum_{i=0}^{[\frac{k+p}{2}]} \frac{b_{i,k} z^{i+1} (xz - y^2)^{k+p-2i}}{i+1}\Big).$$

5. *Poisson bracket:*

$$\mathbf{w} = \sum_{p=1}^{3} \{x_p, \mathbf{I}(x, y, z)\} \cdot \mathbf{e}_p,$$

   where $\mathbf{I}(x, y, z)$ *is the invariant given in Equation* (5.2)*. Furthermore,* $\{\mathbf{I}(x, y, z), \Delta\} = 0$.

Proof. Follow Equation (2.12), Lemma (3.2), item 2 in Theorem (4.3), and Theorem 3.4, respectively. □

The normal form vector field (5.1) generally can not be further simplified. However, the secondary invariant provides a possible reduction in its dimension and then, a further normalization is possible when we consider a subfamily of normal form vector fields given by

$$\mathbf{w} := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + z^p(z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}) + \sum_{i=p+1}^{\infty} b_i z^i (z\frac{\partial}{\partial y} + 2y\frac{\partial}{\partial x}). \tag{5.5}$$

The next theorem deals with the dimension reduction and also a further hypernormalization.

Theorem 5.3. *There exists a near identity transformation so that the vector field* (5.5) *is transformed into the normal form vector field*

$$v := -2Y\frac{\partial}{\partial Z} + \left(-X + \frac{p+2}{p+1}Z^{p+1} + \sum_{i=p+1}^{\infty} \frac{b_i(i+2)}{i+1}Z^{i+1}\right)\frac{\partial}{\partial Y}. \tag{5.6}$$

Furthermore, $X(t) := c$ is always constant. Hence, the normal form system (5.6) *has a Hamiltonian*

$$H(Z,Y) := -\Delta(X,Y,Z) = Y^2 - cZ + \frac{1}{p+1}Z^{p+2} + \sum_{i=p+1}^{\infty} \frac{b_i}{i+1}Z^{i+2}. \tag{5.7}$$

On the invariant manifold $\mathbf{I}(x,y,z) = 0$, the normal form vector field takes a further hypernormalization given by

$$v^{(\infty)} = -2Y\frac{\partial}{\partial Z} + \sum_{i=p}^{\infty} a_i Z^{i+1}\frac{\partial}{\partial Y}, \tag{5.8}$$

where $a_i = 0$ for $i = (p+1)(m+1)$, $m \in \mathbb{N}_0$.

Proof. The key idea is to use the secondary Clebsch potential (5.2) as a near-identity transformation, *i.e.,*

$$(X,Y,Z) := \left(x + \frac{1}{p+1}z^{p+1} + \sum_{i=p+1}^{\infty} \frac{b_i}{i+1}z^{i+1}, y, z\right). \tag{5.9}$$

Hence, $X(t)$ is constant. Then, the normal form vector field (5.8) is given by

$$\mathbf{h} := 2\mathsf{B}_0^1 - c\mathsf{B}_{-1}^{-1} + \frac{(p+2)}{p+1}\mathsf{B}_p^{-1} + \sum_{i=p+1}^{\infty} \frac{(i+2)b_i}{i+1}\mathsf{B}_i^{-1},$$

in terms of notations used in [28]. Hence, the second claim follows [2, Theorem 8.9] (but with A and B interchanged). □

5.1. Truncated normal form coefficients

Consider a cubic-degree truncated triple zero vector field

$$v := -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \sum_{i+j+k=2}^{3} x^i y^j z^k \big(a_{ijk}\frac{\partial}{\partial x} + b_{ijk}\frac{\partial}{\partial y} + c_{ijk}\frac{\partial}{\partial z}\big) \in \mathscr{B}, \tag{5.10}$$

where a_{ijk}, b_{ijk} and $c_{ijk} \in \mathbb{R}$. By Theorem 4.8, $\operatorname{div}(v) = 0$ and $v(\Delta) = 0$. The first one implies

$$\begin{aligned}
c_{002} &= -\frac{1}{2}(a_{101} + b_{011}), & a_{200} &= -b_{110},\\
c_{011} &= -a_{110}, & c_{200} &= 0,\\
c_{020} &= 2b_{110}, & a_{101} &= b_{011},\\
a_{020} &= 2b_{011}, & a_{002} &= 0,\\
c_{101} &= -(2a_{200} + b_{110}), & a_{011} &= 2b_{002},\\
c_{110} &= 2b_{200}, & b_{101} &= 0,
\end{aligned} \tag{5.11}$$

while $v(\Delta) = 0$ concludes

$$\begin{aligned}
c_{111} &= -2(a_{210} + b_{120}), & b_{012} &= -2c_{003},\\
b_{111} &= -2(a_{201} + c_{102}), & a_{012} &= 2b_{003},\\
c_{210} &= 2b_{300}, & a_{120} &= -c_{021},\\
a_{111} &= -2b_{021} - 2c_{012}, & a_{021} &= -4c_{003},\\
b_{210} &= -2a_{300}, & a_{102} &= -b_{012} - 3c_{003},\\
c_{201} &= -3a_{300} - b_{210}, & a_{030} &= 2b_{021},\\
c_{030} &= 2b_{120}, & a_{201} &= -c_{102},\\
a_{210} &= -2(b_{201} + b_{120}), & c_{120} &= -4a_{300}.\\
c_{300} &= a_{003} = 0 & c_{012} &= -2(b_{021} + b_{102})
\end{aligned} \tag{5.12}$$

Hence, the vector field (5.10) can be written as

$$\begin{aligned}
v = {} & \mathsf{B}^1_{0,0} + d^{-1}_{1,0}\mathsf{B}^{-1}_{1,0} + d^0_{1,0}\mathsf{B}^0_{1,0} + d^1_{1,0}\mathsf{B}^1_{1,0} + d^2_{1,0}\mathsf{B}^2_{1,0} + d^3_{1,0}\mathsf{B}^3_{1,0} + d^{-1}_{0,1}\mathsf{B}^{-1}_{0,1} + d^{-1}_{2,0}\mathsf{B}^{-1}_{2,0}\\
& + d^1_{0,1}\mathsf{B}^1_{0,1} + d^0_{0,1}\mathsf{B}^0_{0,1} + d^1_{2,0}\mathsf{B}^1_{2,0} + d^2_{2,0}\mathsf{B}^2_{2,0} + d^3_{2,0}\mathsf{B}^3_{2,0} + d^4_{2,0}\mathsf{B}^4_{2,0} + d^5_{2,0}\mathsf{B}^5_{2,0} + d^0_{2,0}\mathsf{B}^0_{2,0},
\end{aligned} \tag{5.13}$$

where

$$
\begin{aligned}
d_{1,0}^{-1} &= b_{002}, & d_{0,1}^{-1} &= \frac{1}{5}(4b_{102} - b_{021}),\\
d_{2,0}^{1} &= (3b_{021} + 3b_{102}), & d_{2,0}^{2} &= -(c_{021} + c_{102}),\\
d_{1,0}^{1} &= a_{110}, & d_{2,0}^{5} &= -b_{300},\\
d_{1,0}^{0} &= 2b_{011}, & d_{2,0}^{4} &= 3a_{300},\\
d_{0,1}^{1} &= \frac{1}{5}(b_{120} - 4b_{201}), & d_{1,0}^{3} &= -b_{200},\\
d_{1,0}^{2} &= -2b_{110}, & d_{0,1}^{0} &= \frac{1}{5}(c_{021} - 4c_{102}),\\
d_{2,0}^{0} &= -3c_{003}, & d_{2,0}^{3} &= -(3b_{201} + 3b_{120}),\\
d_{2,0}^{-1} &= b_{003}.
\end{aligned}
$$

Proposition 5.4. *The quartic truncated normal form for Equation (5.13) is given by*

$$
\begin{aligned}
w = -x\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial z} + \Big(2y\big((xz - y^2)(b_1^1 z + b_1^0) + z(b_0^3 z^2 + b_0^2 z + b_0^1)\big)\Big)\frac{\partial}{\partial x}\\
+\big(z(xz - y^2)(b_1^1 z + b_1^0) + z^2(b_0^3 z^2 + b_0^2 z + b_0^1)\big)\frac{\partial}{\partial y},
\end{aligned}
$$

whose first integrals are $\Delta = xz - y^2$ *and*

$$
\mathbf{I}(x, y, z) = x - b_1^0 z y^2 + \frac{z^2}{2}(b_1^1 xz - b_1^1 y^2 + 4b_0^1 + 2b_1^0 z^2 + \frac{b_0^3}{2}z^2 + \frac{2b_0^2}{3}z),
$$

where

$$
\begin{aligned}
b_0^1 &= b_{002},\\
b_0^2 &= b_{003} + \frac{b_{002}a_{110}}{2} - \frac{3{b_{011}}^2}{4},\\
b_1^0 &= \frac{{a_{110}}^2}{60} + \frac{b_{110}b_{011}}{5} - \frac{b_{200}b_{002}}{5} + \frac{1}{5}(4b_{102} - b_{021}),\\
b_1^1 &= -\frac{{a_{110}}^3}{378} - \frac{{b_{110}}^2 b_{002}}{7} + \frac{(4c_{102} - c_{021})b_{011}}{15} + \frac{{b_{011}}^2 b_{200}}{21}\\
&\quad + \frac{12(b_{201} + b_{120})b_{002}}{105} + \frac{8b_{110}c_{003}}{21}\\
&\quad + \frac{12(b_{021} + b_{102})a_{110}}{105} - \frac{4b_{200}b_{003}}{7} + \frac{2b_{011}(c_{021} + c_{102})}{35}\\
&\quad - \frac{2b_{200}b_{002}a_{110}}{21} - \frac{2b_{110}b_{011}a_{110}}{63}\\
&\quad + \frac{2}{5}(b_{120} - 4b_{201})b_{002} + \frac{(4b_{102} - b_{021})a_{110}}{15},
\end{aligned}
$$

$$b_0^3 = \frac{2b_{003}a_{110}}{3} - \frac{6b_{110}b_{002}b_{011}}{5} + \frac{4(3b_{021}+3b_{102})b_{002}}{15} - \frac{6b_{200}{b_{002}}^2}{5} + \frac{b_{002}{a_{110}}^2}{10} + 2c_{003}b_{011}.$$

The vector potential normal form for vector field (5.13) *is given by*

$$\left(x + \frac{b_0^1}{2}z^2 + b_1^0\Delta z + \frac{b_0^2}{3}z^3 + \frac{b_1^1}{2}\Delta z^2 + \frac{b_0^3}{4}z^4\right)(z, -2y, x).$$

Proof. The normal form coefficients are derived using an implementation of the formulas in Maple. □

Appendix A

Proof of Theorem 2.7. A polynomial expansion for the left hand side in (2.8) is derived in Equation (2.7) while by using the first equation in (2.5), the right hand side is given by

$$\sum_{p=\max\{\sigma_2,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} C_{p,i,j}^{q_1,q_2} \sum_{n=0}^{p} \eta_n^{2p+|r_2-r_1|,2p-\sigma_2+|r_2-r_1|} x^{p-n} y^{|r_2-r_1|} z^{p-\sigma_2-n} \Delta^{s_1+s_2-p+n+\lfloor\frac{r_1+r_2}{2}\rfloor}.$$

Given Lemma 2.5, we remark that $\mathrm{N}^{2p+|r_2-r_1|}(z^{2p-\sigma_2+|r_2-r_1|}) = 0$ for $p < \sigma_2$.

When $r_1 r_2 = 0$, Equation (2.8) is equivalent with the following polynomial equation

$$\begin{aligned}
&\sum_{n=0}^{\min\{s_1+s_2-\sigma_2,s_1+s_2\}} \tilde{\eta}_{n,i,j}^{q_1,q_2} x^{s_1+s_2-n} z^{i+j-(s_1+s_2+r_1+r_2+n)} \Delta^n \\
&= \sum_{n=0}^{\min\{s_1+s_2-\sigma_2,s_1+s_2\}} \\
&\quad \times \sum_{p=s_1+s_2-n}^{s_1+s_2} C_{p,i,j}^{q_1,q_2} \eta_{n+p-(s_1+s_2)}^{2p+|r_2-r_1|,2p+i+j-2(s_1+s_2)} x^{s_1+s_2-n} z^{i+j-(s_1+s_2+r_1+r_2+n)} \Delta^n.
\end{aligned}$$

Hence for each i, j, s_1, s_2, and $0 \le n \le \min\{s_1+s_2-\sigma_2, s_1+s_2\}$, we have

$$\tilde{\eta}_{n,i,j}^{q_1,q_2} = \sum_{k=0}^{n} \eta_{n-k}^{q_1+q_2-2k,i+j-2k} C_{s_1+s_2-k,i,j}^{q_1,q_2}.$$

These introduce a family of upper triangular linear matrix equations. The determinant of the coefficient matrix is given by

$$\prod_{n=0}^{\min\{s_1+s_2-\sigma_2,s_1+s_2\}} \eta_0^{q_1+q_2-2n,i+j-2n} \neq 0.$$

These together with the Equation (2.6) give rise to

$$C^{q_1,q_2}_{p,i,j} = \sum_{r=0}^{s_1+s_2-p} \frac{\tilde{\eta}^{q_1,q_2}_{r,i,j}(p+1)_1^{s_1+s_2-p-r+}(p-\sigma_2+1-|r_2+r_1|)_1^{s_1+s_2-p-r}}{\eta_0^{2p+|r_2-r_1|,2p-\sigma_2}(s_1+s_2-p-r)!2^{p+r-(s_1+s_2)}(4p-2\sigma_2+3)_2^{s_1+s_2-p-r}}.$$

Now let $r_1 = r_2 = 1$. By substituting $y^2 = xz - \Delta$ into Equation (2.7), $\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j)$ is given by

$$\begin{aligned}\mathrm{N}^{q_1}(z^i)\mathrm{N}^{q_2}(z^j) = &\sum_{n=1}^{\min\{s_1+s_2-\sigma_2,s_1+s_2\}} (\tilde{\eta}^{q_1,q_2}_{n,i,j} - \tilde{\eta}^{q_1,q_2}_{n-1,i,j})x^{s_1+s_2-n+1}z^{i+j-s_1-s_2-n-1}\Delta^n \\ &+ \tilde{\eta}^{q_1,q_2}_{0,i,j}x^{s_1+s_2+1}z^{i+j-s_1-s_2-1} \\ &- \tilde{\eta}^{q_1,q_2}_{\min\{s_1+s_2-\sigma_2,s_1+s_2\},i,j}x^{\max\{0,\sigma_2\}}z^{\max\{-\sigma_2,0\}}\Delta^{s_1+s_2+1}.\end{aligned}$$

Hence the family of linear equations and its solutions are derived by

$$\tilde{\eta}^{q_1,q_2}_{n,i,j} - \tilde{\eta}^{q_1,q_2}_{n-1,i,j} = \sum_{k=0}^{n} \eta^{q_1+q_2-2k,i+j-2k}_{n-k} C^{q_1,q_2}_{s_1+s_2-k,i,j},$$

and

$$C^{q_1,q_2}_{n,i,j} := \sum_{r=0}^{s_1+s_2+1-n} \frac{\left(\tilde{\eta}^{q_1,q_2}_{r,i,j} - \tilde{\eta}^{q_1,q_2}_{r-1,i,j}\right)(n+1)_1^{s_1+s_2-n-r+1}(n-\sigma_2-1)_1^{s_1+s_2-n-r+1}}{(s_1+s_2-n-r)!2^{n+r-(s_1+s_2+1)}\eta_0^{2n,2n-\sigma_2}(4n-2\sigma_2-1)_2^{s_1+s_2-n-r+1}},$$

respectively. □

Appendix B

The following notations are used in Theorem 2.10:

$$l^{q_1,q_2}_{1,3,i_1,i_2} := \frac{-(2i_1-q_1+1)(q_2-1)_1^3}{(i_1+1)\kappa_{q_2,2i_2+2}\kappa_{q_1+2,2i_1+2}},$$

$$l^{q_1,q_2}_{2,3,i_1,i_2} := \frac{-2(q_2)_1^2(i_1-q_1)}{(i_1+1)\kappa_{q_2,2i_2+2}\kappa_{q_1+1,2i_1+2}},$$

$$l^{q_1,q_2}_{3,3,i_1,i_2} := \frac{(q_1+1)(q_2+1)}{(i_1+1)\kappa_{q_2,2i_2+2}\kappa_{q_1,2i_1+2}},$$

$$l^{q_1,q_2}_{1,2,i_1,i_2} := \frac{-(q_2)_1^2(2i_1-q_1+1)(i_2-q_2)}{(i_1+1)\kappa_{q_2+1,2i_2+2}\kappa_{q_1+2,2i_1+2}},$$

$$l^{q_1,q_2}_{2,2,i_1,i_2} := \frac{-2q_2(i_1-q_1)(i_2-q_2)}{(i_1+1)\kappa_{q_2+1,2i_2+2}\kappa_{q_1+1,2i_1+2}},$$

$$l^{q_1,q_2}_{3,2,i_1,i_2} := \frac{(q_1+1)(i_2-q_2)}{(i_1+1)\kappa_{q_2+1,2i_2+2}\kappa_{q_1,2i_1+2}}. \tag{B.1}$$

Now we present the proof of Theorem 2.10.

Proof of Theorem *2.10*. By equations (2.12), (2.8), and Lemma 2.4, the third component of $\mathsf{B}^{q_1}_{i_1,k_1}\mathsf{B}^{q_2}_{i_2,k_2}$ is given by

$$\begin{aligned}
&\sum_{p=1}^{3} l^{q_1,q_2}_{p,3,i_1,i_2}\Delta^{k_1+k_2}\mathrm{N}^{q_1+3-p}(z^{i_1+1})\mathrm{N}^{q_2-3+p}(z^{i_2})\\
&= \sum_{j=\max\{\sigma_2-1,0\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor}\sum_{p=1}^{3}\frac{l^{q_1,q_2}_{p,3,i_1,i_2}C^{q_1+3-p,q_2-3+p}_{j,i_1+1,i_2}\mathrm{N}^{2j+|r_2-r_1|}(z^{2j-\sigma_2+1+|r_2-r_1|})}{\Delta^{j-\sigma_1-\lfloor\frac{r_1+r_2}{2}\rfloor}},
\end{aligned}$$

where $l^{q_1,q_2}_{p,3,i_1,i_2}$ for $p=1,2,3$ are defined by equations (B.1). Now using the latter and Equation (2.12), the third component of $[\mathsf{B}^{q_1}_{i_1,k_1},\mathsf{B}^{q_2}_{i_2,k_2}] = \mathsf{B}^{q_1}_{i_1,k_1}\mathsf{B}^{q_2}_{i_2,k_2} - \mathsf{B}^{q_2}_{i_2,k_2}\mathsf{B}^{q_1}_{i_1,k_1}$ is given by

$$\begin{aligned}
&\sum_{j=\max\{\sigma_2-1,-1\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor}\frac{\sum_{p=1}^{3} l^{q_2,q_1}_{p,3,i_2,i_1}C^{q_2+3-p,q_1-3+p}_{j,i_2+1,i_1} - \sum_{p=1}^{3} l^{q_1,q_2}_{p,3,i_1,i_2}C^{q_1+3-p,q_2-3+p}_{j,i_1+1,i_2}}{(2j+|r_2-r_1|+1)(2j+1-\sigma_2+|r_2-r_1|)^{-1}(\kappa_{2j+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|})^{-1}}\\
&\times\mathsf{B}^{2j+|r_2-r_1|}_{2j-\sigma_2+|r_2-r_1|,\sigma_1-j+\lfloor\frac{r_1+r_2}{2}\rfloor}\cdot\mathbf{e}_3,
\end{aligned}$$

when $2j+|r_2-r_1|+1\neq 0$. The second component of $\mathsf{B}^{q_1}_{i,k_1}\mathsf{B}^{q_2}_{j,k_2}$ is given by

$$\begin{aligned}
&\sum_{p=1}^{3} l^{q_1,q_2}_{p,2,i_1,i_2}\Delta^{k_1+k_2}\mathrm{N}^{q_1+3-p}(z^{i_1+1})\mathrm{N}^{q_2-2+p}(z^{i_2})\\
&= \sum_{j=\max\{\sigma_2-1,-1\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor}\sum_{p=1}^{3} l^{q_1,q_2}_{p,2,i_1,i_2}C^{q_1+3-p,q_2-2+p}_{j,i_1+1,i_2}\mathrm{N}^{2j+|r_2-r_1|+1}(z^{2j-\sigma_2+|r_2-r_1|+1})\Delta^{\sigma_1-j+\lfloor\frac{r_1+r_2}{2}\rfloor},
\end{aligned}$$

where the constants $l^{q_1,q_2}_{p,2,i_1,i_2}$ are defined by equations (B.1). Again through the Equation (2.12), the second component of $[\mathsf{B}^{q_1}_{i_1,k_1},\mathsf{B}^{q_2}_{i_2,k_2}]$ is given by

$$\begin{aligned}
&\sum_{j=\max\{\sigma_2-1,-1\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor}\frac{\sum_{p=1}^{3} l^{q_1,q_2}_{p,2,i_1,i_2}C^{q_1+3-p,q_2-2+p}_{j+|r_2-r_1|,i_1+1,i_2} - \sum_{p=1}^{3} l^{q_2,q_1}_{p,2,i_2,i_1}C^{q_2+3-p,q_1-2+p}_{j+|r_2-r_1|,i_2+1,i_1}}{\sigma_2(2j+1-\sigma_2+|r_2-r_1|)^{-1}(\kappa_{2j+1+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|})^{-1}}\\
&\times\mathsf{B}^{2j+|r_2-r_1|}_{2j-\sigma_2+|r_2-r_1|,\sigma_1-j+\lfloor\frac{r_1+r_2}{2}\rfloor}\cdot\mathbf{e}_2.
\end{aligned} \tag{B.2}$$

On the other hand when $2j+1+|r_2-r_1|\neq 0$

$$b^{q_1,q_2}_{j,i_1,i_2} = \frac{\sum_{p=1}^{3} l^{q_1,q_2}_{p,2,i_1,i_2} C^{q_1+3-p,q_2-2+p}_{j+|r_2-r_1|,i_1+1,i_2} - \sum_{p=1}^{3} l^{q_2,q_1}_{p,2,i_2,i_1} C^{q_2+3-p,q_1-2+p}_{j+|r_2-r_1|,i_2+1,i_1}}{\sigma_2(2j+1-\sigma_2+|r_2-r_1|)^{-1}(\kappa_{2j+1+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|})^{-1}}.$$

We remark that the third component of $\mathsf{B}^{-1}_{-\sigma_2-1,\sigma_1+1}$ is always zero and this corresponds to the condition $2j+1+|r_2-r_1|=0$. Indeed, this condition occurs when $j=-1$ and

$$(q_1 := 2s_1 \text{ and } q_2 := 2s_2+1) \quad \text{or} \quad (q_1 := 2s_1+1 \text{ and } q_2 := 2s_2).$$

Hence, the constant a^{q_1,q_2}_{-1,i_1,i_2} are derived through Equation (B.2). Now the proof is complete by derivation of the formula for the first component of $[\mathsf{B}^{q_1}_{i_1,k_1}, \mathsf{B}^{q_2}_{i_2,k_2}]$ as

$$\sum_{j=\max\{\sigma_2-1,-1\}}^{s_1+s_2+\lfloor\frac{r_1+r_2}{2}\rfloor} \frac{\sum_{p=1}^{3} l^{q_1,q_2}_{p,1,i_1,i_2} C^{q_1+3-p,q_2-1+p}_{j,i_1+1,i_2} - \sum_{p=1}^{3} l^{q_2,q_1}_{p,1,i_2,i_1} C^{q_2+3-p,q_1-1+p}_{j,i_2+1,i_1}}{(2j+1-2\sigma_2+|r_2-r_1|)(2j+1-\sigma_2+|r_2-r_1|)^{-1}(\kappa_{2j+2+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|})^{-1}}$$
$$\times \mathsf{B}^{2j+|r_2-r_1|}_{2j-\sigma_2+|r_2-r_1|,\sigma_1-j+\lfloor\frac{r_1+r_2}{2}\rfloor} \cdot \mathbf{e}_1,$$

where

$$l^{q_1,q_2}_{1,1,i_1,i_2} := \frac{-(q_2+1)^2_1(2i_1-q_1+1)(2i_2-q_2+1)}{(i_1+1)\kappa_{q_2+2,2i_2+2}\kappa_{q_1+2,2i_1+2}},$$
$$l^{q_1,q_2}_{2,1,i_1,i_2} := \frac{-2(q_2+2)(i_1-q_1)(2i_2-q_2+1)}{(i_1+1)\kappa_{q_2+2,2i_2+2}\kappa_{q_1+1,2i_1+2}},$$
$$l^{q_1,q_2}_{3,1,i_1,i_2} := \frac{(q_1+1)(2i_2-q_2+1)}{(i_1+1)q_2+2,2i_2+2\kappa_{q_1,2i_1+2}},$$

and the equality

$$b^{q_1,q_2}_{j,i_1,i_2} = \frac{\sum_{p=1}^{3} l^{q_1,q_2}_{p,1,i_1,i_2} C^{q_1+3-p,q_2-1+p}_{j,i_1+1,i_2} - \sum_{p=1}^{3} l^{q_2,q_1}_{p,1,i_2,i_1} C^{q_2+3-p,q_1-1+p}_{j,i_2+1,i_1}}{(2j+1-2\sigma_2+|r_2-r_1|)(2j+1-\sigma_2+|r_2-r_1|)^{-1}(\kappa_{2j+2+|r_2-r_1|,4j-2\sigma_2+2+2|r_2-r_1|})^{-1}}. \quad \square$$

Appendix C

Definitions of $F_0(n,p)$, $F_1(n,p)$, $G_0(n,p)$, and $G_1(n,p)$ in Theorem 4.3 are given by:

$$F_0(n,p) = \frac{(2s)!i!(i+1)!(2s+1)!2^{2n+2}}{(2p)!(s-p)!(i-p-s)!(s-n+p-1)!(2n-2p)!(i+p-n-s-1)!}$$
$$\times\left(\frac{(s+1)(i+p-n-s)^{-1}}{(s-p+1)(2n-2p+1)} + \frac{(i-2s)(i+p-n-s)^{-1}}{(2p+1)(s+p-n)}\right.$$
$$\left. - \frac{(i-s+1)(i-p-s+1)^{-1}}{(2n-2p+1)(s+p-n)}\right),$$

$$F_1(n,p) = \frac{2^{2n+1} i!(i+1)!(2s+1)!(2s+2)!}{(s-n+p)!(2p)!(s-p)!(2n-2p)!(i-p-s-1)!(i-n+p-s-1)!}$$
$$\times\Big(\frac{(2s+3)(i-n+p-s)^{-1}}{(s+1-p)(2p+1)} + \frac{2(i-2s-1)(i-p-s)^{-1}}{(s+1-p)(2n-2p+1)}$$
$$-\frac{(2i+1-2s)(i-p-s)^{-1}}{(2p+1)(s+1-n+p)}\Big),$$

$$G_0(n,p) = \frac{p\,(2s-i)\left(2ip+2s^2+5p-2is-2i-2n-5\right)}{(2n-2p+3)\,(n-p+1)\,(2p)!}$$
$$\times\frac{1}{(s-p+1)!\,(2n-2p+1)!\,(i-p-s+1)!\,(s+p-n-1)!\,(i+p-n-s-1)!},$$

$$G_1(n,p) = \frac{p2^{2n+2}(2s+3)(2i-2s+1)(i-2s-1)}{(i-p-s)(2n-2p+1)(s+1-p)(n-p+1)(3+2n-2p)}$$
$$\times\frac{2i!(i+1)!(2s+1)!(2s+2)!}{(i-p-s-1)!(2n-2p)!(s-p)!(2p)!(i-n+p-s-1)!(s-n+p)!}.$$

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